PROBLEMS IN STRENGTH OF MATERIALS

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PROBLEMS IN

STRENGTH OF MATERIALS

(*VOLUME II*)

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Dedicated to my

mother Sreemati Lilabati Bose

without whose loving care

I would not have been what I am

PREFACE

The second volume of the book entitled "Problems in Strength of Materials" is being published to meet the needs of students of higher classes in Engineering Colleges. Some of the chapters will also be useful to candidates appearing in Engineering Services and Associate Membership of the Institution of Engineers examinations.

The majority of the problems, dealt with here; have been taken from the examination papers of the London University. And a few of these problems were set in the examinations conducted by the Union Public Service Commission and the Institution of Engineers. The author wishes to thank these bodies for their kind permission to reproduce the problems set by them. Many of the problems, however, were originally set in F. P. S. units; they have now been converted to M. K. S. units.

A glossary of notations used in the book is included, and the readers should consult it in case they have any doubt.

The author has already received many encouraging letters from the learned professors of different technical institutions in India appreciating the first volume of this book. He has also received many valuable suggestions for the improvement of the book. The author takes this opportunity to express his profound gratitude to all these well-wishers whom he has not yet been able to thank personally. He will consider their suggestions while publishing the second edition of the first volume.

The author wishes to express his deep sense of gratitude to the Publisher of the book for the care that he took in producing the book.

While every effort has been made to check the solutions of the problems, the author would be grateful to his readers for indicating to him whatever errors and misprints that might have still remained.

Patna, **B. N. Bose**
March 3, 1973.

CONTENTS

NOTATIONS

A	Area
A'	Modified area
B	Bending moment
b	Width
C	Modulus of rigidity
D	Diameter
d	Depth, diameter
E	Modulus of elasticity
e	Direct strain, eccentricity
F	Shearing force
f	Direct stress
g	Acceleration due to gravity
h	Height
I	Moment of inertia
J	Polar moment of inertia
K	Bulk modulus, stiffness
k	Radius of gyration
L, l	Length
M	Bending moment
$\frac{1}{m}$	Poisson's ratio
N	Frequency of vibration, number of revolutions
P	Force
p	Pitch, pressure
Q	Force
q	Shearing stress
R	Radius, reaction
r	Radius

T	Temperature, time, torque
t	Thickness
U	Strain energy
V	Volume
v	Velocity
W	Load
w	Distributed load, weight per unit length
x	Distance
y	Deflection
Z	Section modulus
α	Angle, coefficient of thermal expansion
β	Angle
δ	Deflection, elongation
θ	Angle of twist, slope of beam.
μ	Coefficient of friction
ρ	Density
φ	Angle
ω	Angular velocity

CHAPTER I

BENDING COMBINED WITH TORSION AND THRUST

1. What must be the diameter of a solid shaft to transmit a twisting moment of 60 tonne metres and a bending moment of 15 tonne metres, the maximum direct stress being limited to 800 kg/cm² ? What should be the external diameter of a hollow shaft to do this if the internal diameter is 0·6 of the external diameter ? (Engineering Services, 1967)

Equivalent bending moment,

$$M_e = \tfrac{1}{2}[M + \sqrt{M^2 + T^2}] = \tfrac{1}{2}[15 + \sqrt{15^2 + 60^2}]$$
$$= 38{\cdot}4 \text{ t m} = 38{\cdot}4 \times 10^5 \text{ kg cm}$$

But $$M_e = f.Z = \frac{\pi}{32} D^3 . f$$

$\therefore$ $$\frac{\pi}{32} D^3 \times 800 = 38{\cdot}4 \times 10^5$$

or $$D^3 = \frac{38{\cdot}4 \times 10^5 \times 32}{\pi \times 800}$$

Solving $$D = \mathbf{36{\cdot}6 \text{ cm}}$$

For the hollow shaft,

$$I = \frac{\pi}{64}[D^4 - d^4] = \frac{\pi}{64}[D^4 - (0{\cdot}6D)^4] = 0{\cdot}0427 D^4$$

$$M_e = f.Z = \frac{800 \times 0{\cdot}0427 D^4 \times 2}{D}$$

$$\therefore \quad \frac{800 \times 0{\cdot}0427 D^4 \times 2}{D} = 38{\cdot}4 \times 10^5$$

or $$D^3 = \frac{38{\cdot}4 \times 10^5}{800 \times 0{\cdot}0427 \times 2}$$

Solving $\quad D = \mathbf{38{\cdot}3 \text{ cm}}$.

2. In a circular shaft subjected to an axial twisting moment T and a bending moment M, show that when $M = 1{\cdot}2T$, the ratio of the maximum shearing stress to the greater principal stress is approximately 0·566. (Lond. Univ.)

Equivalent bending moment,

$$M_e = \tfrac{1}{2}[M + \sqrt{M^2 + T^2}] = \tfrac{1}{2}[1{\cdot}2T + \sqrt{(1{\cdot}2T)^2 + T^2}]$$
$$= \tfrac{1}{2}[1{\cdot}2T + 1{\cdot}562T] = 1{\cdot}381T$$

Maximum principal stress,

$$f_1 = \frac{M_e}{Z} = \frac{32M_e}{\pi D^3} = \frac{32 \times 1{\cdot}381T}{\pi D^3}$$

Equivalent torque, $T_e = \sqrt{M^2 + T^2} = 1{\cdot}562T$

Maximum shear stress,

$$q_{max} = \frac{T_e}{J} \times \frac{D}{2} = \frac{16T_e}{\pi D^3} = \frac{16 \times 1{\cdot}562T}{\pi D^3}$$

$$\therefore \quad \frac{q_{max}}{f_1} = \frac{16 \times 1{\cdot}562}{32 \times 1{\cdot}381} = 0{\cdot}566.$$

3. A hollow circular shaft 20 cm external diameter and 10 cm internal diameter is subjected to a direct compression load of 75 tonnes, a bending moment of 450 tonnes cm and a twisting moment of 620 tonnes cm. Calculate the maximum principal stress and the maximum shearing stress.
(Lond. Univ.)

$$A = \frac{\pi}{4}(20^2 - 10^2) = 236 \text{ cm}^2$$

$$I = \frac{\pi}{64}(20^4 - 10^4) = 7{,}360 \text{ cm}^4$$

Stress due to end thrust

$$= \frac{P}{A} = \frac{75{,}000}{236} = 318 \text{ kg/cm}^2$$

Bending stress $= \dfrac{M}{Z} = \dfrac{450{,}000 \times 10}{7{,}360} = 611 \text{ kg/cm}^2$

Total direct stress, $f = 318 + 611 = 929 \text{ kg/cm}^2$

Shear stress, $q = \dfrac{T}{J} \times \dfrac{D}{2} = \dfrac{620{,}000 \times 20}{2 \times 7{,}360 \times 2} = 421 \text{ kg/cm}^2$

Maximum principal stress,

$$\begin{aligned} f_1 &= \tfrac{1}{2}f + \tfrac{1}{2}\sqrt{f^2 + 4q^2} \\ &= \tfrac{1}{2} \times 929 + \tfrac{1}{2}\sqrt{929^2 + 4 \times 421^2} \\ &= 464{\cdot}5 + 627 = \mathbf{1091{\cdot}5 \text{ kg/cm}^2} \end{aligned}$$

Maximum shearing stress,

$$\begin{aligned} q_{max} &= \tfrac{1}{2}(f_1 - f_2) = \tfrac{1}{2}\sqrt{f^2 + 4q^2} \\ &= \mathbf{627 \text{ kg/cm}^2}. \end{aligned}$$

4. A flywheel weighing 600 kg is mounted on a shaft 8 cm in diameter and midway between bearings 60 cm apart, in which the shaft may be assumed to be directionally free. If the shaft is transmitting 40 horse-power at 360 r. p. m., calculate the principal stresses and the maximum shearing stresses in the shaft at the ends of a vertical and a horizontal diameter in a plane close to that of the flywheel. (Engineering Services, 1968)

Consider a section close to the flywheel.

Bending moment, $M=\dfrac{WL}{4}=\dfrac{600\times 60}{4}=9{,}000$ kg cm

Shearing force, $F=\dfrac{W}{2}=300$ kg

$$\text{Horse-power}=40=\frac{2\pi\times 360T}{4{,}500}$$

$$\therefore\quad T=79{\cdot}6 \text{ kg m}$$

(a) Consider the ends of the vertical diameter.

Stress due to bending,

$$f=\frac{32M}{\pi D^3}=\frac{32\times 9{,}000}{\pi\times 8^3}=179 \text{ kg/cm}^2$$

Shear stress due to torque,

$$q=\frac{16T}{\pi D^3}=\frac{16\times 7{,}960}{\pi\times 8^3}=79{\cdot}2 \text{ kg/cm}^2$$

Shear stress due to F is zero.

$$\text{Principal stresses}=\tfrac{1}{2}f\pm\tfrac{1}{2}\sqrt{f^2+4q^2}$$

$$=\tfrac{1}{2}\times 179\pm\tfrac{1}{2}\sqrt{179^2+4\times 79{\cdot}2^2}$$

$$=\mathbf{209\ kg/cm^2}\text{ and }\mathbf{-30\ kg/cm^2}$$

Maximum shear stress,

$$q_{max}=\tfrac{1}{2}(f_1-f_2)=\tfrac{1}{2}(209+30)=\mathbf{119{\cdot}5\ kg/cm^2}.$$

(b) Consider the ends of the horizontal diameter.

Stress due to $\quad M=0$

Shear stress due to $T=79{\cdot}2$ kg/cm²

Shear stress due to F

$$=\frac{4}{3}\times\frac{F}{A}=\frac{4\times 300\times 4}{3\times\pi\times 8^2}=7{\cdot}96 \text{ kg/cm}^2$$

Maximum shear stress,

$$q_{max}=79{\cdot}2+7{\cdot}96=\mathbf{87{\cdot}16\ kg/cm^2}$$

Principal stresses $=\pm q_{max}=\pm\,\mathbf{87{\cdot}16\ kg/cm^2}.$

5. A hollow shaft is 16 cm external diameter and 8 cm internal diameter. It transmits 475 h. p. at 220 r. p. m. What bending moment could be carried in addition if the maximum shearing stress must not exceed 350 kg/cm² ?

(Lond. Univ.)

$$J=\frac{\pi}{32}(16^4-8^4)=6{,}030 \text{ cm}^4$$

$$\text{H. P.}=475=\frac{2\pi\times 220T}{4{,}500}$$

$$\therefore \quad T=1{,}546 \text{ kg m}$$

$$T_e=\frac{2q}{D}\times J=\frac{2\times 350\times 6{,}030}{16}$$

$$=264{,}000 \text{ kg cm}$$

$$=2{,}640 \text{ kg m}$$

But $\quad T_e=\sqrt{M^2+1{,}546^2}$

$\therefore \quad \sqrt{M^2+1{,}546^2}=2{,}640$

Solving $\quad M=$ **2,140 kg m.**

6. A shaft of 20 cm diameter transmits 2,500 h. p. at 250 r. p. m. and is subjected to a bending moment of 500 tonnes cm. Calculate the maximum permissible end thrust on the shaft if the maximum shearing stress must not exceed 800 kg/cm².

(Lond. Univ.)

$$\text{H. P.}=2{,}500=\frac{2\pi\times 250T}{4{,}500}$$

$$\therefore \quad T=7{,}160 \text{ kg m}$$

Shear stress due to T,

$$q=\frac{16T}{\pi D^3}=\frac{16\times 716{,}000}{\pi\times 20^3}$$

$$=456 \text{ kg/cm}^2$$

Let f be the total direct stress.

Then maximum shear stress

$$=800=\tfrac{1}{2}\sqrt{f^2+4q^2}$$

$$=\tfrac{1}{2}\sqrt{f^2+4\times 456^2}$$

Rearranging and squaring

$$1{,}600^2=f^2+4\times 456^2$$

$$\therefore \quad f=1{,}315 \text{ kg/cm}^2$$

Stress due to bending

$$=\frac{32M}{\pi D^3}=\frac{32\times 500{,}000}{\pi\times 20^3}=637\ \text{kg/cm}^2$$

$\therefore$ Stress due to end thrust

$$=1{,}315-637=678\ \text{kg/cm}^2$$

End thrust $\qquad P=\frac{\pi}{4}\times 20^2\times 678=213{,}000$ kg

$$=\mathbf{213\ tonnes.}$$

7. A hollow steel shaft 9 cm external diameter and 2·5 cm thick transmits 300 h.p. at 250 r. p. m. and is subjected to an axial thrust of 5 tonnes in addition to a bending moment M. Determine the value of M if the greater principal stress is limited to 900 kg/cm². What is then the value of the smaller principal stress? (Lond. Univ.)

$$A=\frac{\pi}{4}(9^2-4^2)=51{\cdot}1\ \text{cm}^2$$

$$I=\frac{\pi}{64}(9^4-4^4)=309\ \text{cm}^4$$

$$\text{H. P.}=300=\frac{2\pi\times 250T}{4{,}500}$$

$$\therefore\quad T=859\ \text{kg m}$$

Shear stress due to T,

$$q=\frac{T}{J}\times\frac{D}{2}=\frac{85{,}900\times 9}{2\times 309\times 2}=625\ \text{kg/cm}^2$$

Let f be the total direct stress.

Maximum principal stress,

$$f_1=900=\tfrac{1}{2}f+\tfrac{1}{2}\sqrt{f^2+4\times 625^2}$$

Rearranging and squaring

$$(1{,}800-f)^2=f^2+4\times 625^2$$

$$\therefore\quad f=466\ \text{kg/cm}^2$$

$$\text{Stress due to axial thrust}=\frac{P}{A}=\frac{5{,}000}{51{\cdot}1}$$

$$=97{\cdot}8\ \text{kg/cm}^2$$

$\therefore$ Stress due to bending $= 466 - 97{\cdot}8$

$$= 368{\cdot}2 \text{ kg/cm}^2$$

Bending moment, $M = \dfrac{368{\cdot}2 \times 309}{4{\cdot}5}$

$$= \mathbf{25{,}300 \text{ kg cm}}$$

Let f_2 be the smaller principal stress.

Then $$f_1 + f_2 = f$$

$$\therefore \quad f_2 = f - f_1 = 466 - 900$$

$$= \mathbf{-434 \text{ kg/cm}^2}$$

Negative sign indicates tensile stress.

8. If a shaft having a diameter of 10 cm is subjected to a bending moment of 65,000 kg cm, in addition to the torque which it transmits, find the maximum torque allowable if the direct stress in the shaft is not to exceed 800 kg/cm² and the shearing stress is not to exceed 550 kg/cm². State clearly which of the two limiting stresses is reached and determine the maximum value of the other stress. (Lond. Univ.)

$$M_e = \frac{\pi}{32} D^3 . f = \frac{\pi}{32} \times 10^3 \times 800$$

$$= 78{,}500 \text{ kg cm}$$

But $$M_e = \tfrac{1}{2}[65{,}000 + \sqrt{65{,}000^2 + T^2}]$$

$$\therefore \quad \tfrac{1}{2}[65{,}000 + \sqrt{65{,}000^2 + T^2}] = 78{,}500$$

Rearranging and squaring,

$$65{,}000^2 + T^2 = 92{,}000^2$$

Solving $$T = 65{,}100 \text{ kg cm}$$

$$T_e = \frac{\pi}{16} D^3 . q = \frac{\pi}{16} \times 10^3 \times 550$$

$$= 108{,}000 \text{ kg cm}$$

But $$T_e = \sqrt{65{,}000^2 + T^2}$$

$$\therefore \quad \sqrt{65{,}000^2 + T^2} = 108{,}000$$

Squaring $$65{,}000^2 + T^2 = 108{,}000^2$$

Solving $$T = 86{,}200 \text{ kg cm}$$

Hence maximum allowable torque

$= \mathbf{65{,}100 \text{ kg cm}}$ from direct stress criterion.

Equivalent torque $= \sqrt{65{,}000^2 + 65{,}100^2} = 92{,}000$ kg cm

Shearing stress $= \dfrac{16 \times 92{,}000}{\pi \times 10^3} = \mathbf{469 \text{ kg/cm}^2}$.

9. At a point on the surface of a solid circular shaft of 15 cm diameter subjected to combined bending and torsion, the principal stresses are 1,200 kg/cm² tension and 400 kg/cm² compression. Find the bending moment and torque.

If the maximum shearing stress in the material is limited to 1,000 kg/cm², find by how much the torque can be increased, the bending moment remaining constant. (Lond. Univ.)

We know that $\frac{1}{2}[M \pm \sqrt{M^2+T^2}]=\frac{\pi}{32}D^3.f$

Hence $\frac{1}{2}[M+\sqrt{M^2+T^2}]=\frac{\pi}{32}\times 15^3\times 1{,}200$

or $M+\sqrt{M^2+T^2}=795{,}000$ kg cm .. (1)

and $\frac{1}{2}[M-\sqrt{M^2+T^2}]=-\frac{\pi}{32}\times 15^3\times 400$

or $M-\sqrt{M^2+T^2}=-265{,}000$ kg cm .. (2)

Adding equations 1 and 2

$2M=530{,}000 \quad \therefore \quad M=$ **265,000 kg cm**

From equation 1

$265{,}000^2+T^2=(795{,}000-265{,}000)^2$

Solving $T=$ **459,000 kg cm**

Equivalent torque, T_e

$=\frac{\pi}{16}D^3.q=\frac{\pi}{16}\times 15^3\times 1{,}000$

$=663{,}000$ kg/cm

But $T_e=\sqrt{265{,}000^2+T^2}$

$\therefore \quad \sqrt{265{,}000^2+T^2}=663{,}000$

Squaring $265{,}000^2+T^2=663{,}000^2$

$\therefore \quad T=608{,}000$ kg cm

$\therefore$ Increase in torque

$=608{,}000-459{,}000=$ **149,000 kg cm.**

10. Find the dimensions of a hollow steel shaft, internal diameter = 0·6 × external diameter, to transmit 200 horse-power at a speed of 250 r. p. m. if the shearing stress is not to exceed 700 kg/cm².

If a bending moment of 300 kg metre is now applied to the shaft find the speed at which it must be driven to transmit the same horse-power for the same value of the maximum shearing stress. (Lond. Univ.)

$$J=\frac{\pi}{32}[D^4-(0{\cdot}6D)^4]=0{\cdot}0855\,D^4$$

(a) $$\text{H. P.}=200=\frac{2\pi\times 250T_1}{4{,}500}$$

$\therefore\quad T_1=573$ kg m

But $$T_1=\frac{2q}{D}\times J$$

$$\therefore\quad \frac{2\times 700\times 0{\cdot}0855D^4}{D}=57{,}300$$

or $$D^3=\frac{57{,}300}{2\times 700\times 0{\cdot}0855}$$

Solving $D=7{\cdot}82$ cm

$d=0{\cdot}6D=4{\cdot}69$ cm.

(b) Equivalent torque, T_e

$$=573=\sqrt{M^2+T_2^2}$$

$$=\sqrt{300^2+T_2^2}$$

or $$300^2+T_2^2=573^2$$

Solving $T_2=488$ kg m

For transmission of the same horse-power

$$N_1T_1=N_2T_2$$

or $$250\times 573=N_2\times 488$$

$\therefore\quad N_2=$ **294** r. p. m.

11. A shaft 12 cm diameter is subjected to a thrust of 15 tonnes along its axis. There is a bending moment on the shaft equal to half the twisting moment. If the maximum stress is 900 kg/cm², find the H. P. which can be transmitted at 120 r. p. m.

$$M=\tfrac{1}{2}T$$

Stress due to thrust $=\dfrac{15{,}000\times 4}{\pi\times 12^2}=133$ kg/cm²

Stress due to bending moment,

$$f_b=\frac{32M}{\pi D^3}=\frac{32\times\dfrac{T}{2}}{\pi D^3}=\frac{16T}{\pi D^3}$$

Total direct stress, $f=133+f_b$

Shear stress due to torque,

$$q = \frac{16T}{\pi D^3} = f_b$$

Maximum principal stress

$$= 900 = \tfrac{1}{2}(133 + f_b) + \tfrac{1}{2}\sqrt{(133 + f_b)^2 + 4f_b^2}$$

Rearranging and squaring

$$(1{,}667 - f_b)^2 = (133 + f_b)^2 + 4f_b^2$$

or $$4f_b^2 + 3{,}600f_b - 2{,}761{,}000 = 0$$

Solving $$f_b = 495 \text{ kg/cm}^2$$

$$T = \frac{\pi D^3}{16} \times f_b = \frac{\pi \times 12^3 \times 495}{16}$$

$$= 167{,}900 \text{ kg cm}$$

$$= 1{,}679 \text{ kg m}$$

$$\text{H. P.} = \frac{2\pi \times 120 \times 1{,}679}{4{,}500} = \mathbf{281.}$$

12. A steel shaft, supported in bearings A and B at its ends, carries a pulley at C as shown in Fig. 1. Power is applied by a torque T at A and taken off through a belt overrunning the pulley, the tensions in the two branches of the belt being as shown. The allowable stresses for the shaft in tension and shear are 800 kg/cm² and 400 kg/cm². Find the diameter of the shaft.

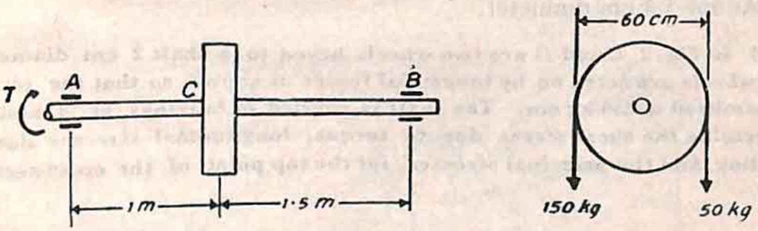

Fig. 1

Between A and C torque,

$$T = (150 - 50)30 = 3{,}000 \text{ kg cm}$$

Maximum bending moment at C

$$M = \frac{(150 + 50)1{\cdot}5 \times 1}{2{\cdot}5} = 120 \text{ kg m}$$

$$= 12{,}000 \text{ kg cm}$$

The weakest cross-section is C.

$$M_e=\tfrac{1}{2}[12{,}000+\sqrt{12{,}000^2+3{,}000^2}\,]$$
$$=\tfrac{1}{2}[12{,}000+12{,}370]$$
$$=12{,}185 \text{ kg cm}$$

But $$M_e=\frac{\pi}{32}D^3\times 800$$

$$\therefore\quad \frac{\pi}{32}D^3\times 800=12{,}185$$

or $$D^3=\frac{12{,}185\times 32}{\pi\times 800}$$

Solving $$D=5{\cdot}37 \text{ cm}$$

$$T_e=\sqrt{12{,}000^2+3{,}000^2}$$
$$=12{,}370$$

But $$T_e=\frac{\pi}{16}D^3\times 400$$

$$\therefore\quad \frac{\pi}{16}D^3\times 400=12{,}370$$

or $$D^3=\frac{12{,}370\times 16}{\pi\times 400}$$

Solving $$D=\mathbf{5{\cdot}40} \text{ cm}$$

Adopt 5·4 cm diameter.

13. In Fig. 2, C and D are two wheels keyed to a shaft 2 cm diameter The wheels are acted on by tangential forces as shown, so that the torque transmitted is 250 kg cm. The shaft is carried in bearings at A and B. Determine the shear stress due to torque, longitudinal stresses due to bending, and the principal stresses, for the top point of the cross-section at P.

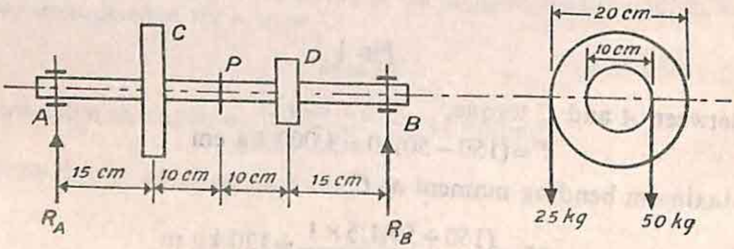

Fig. 2

Taking moment about B

$$R_A\times 50=25\times 35+50\times 15$$

$$\therefore \quad R_A = 32{\cdot}5 \text{ kg}$$

$$R_A + R_B = 25 + 50 = 75 \text{ kg}$$

$$\therefore \quad R_B = 42{\cdot}5 \text{ kg}$$

Bending moment at P,

$$M = 32{\cdot}5 \times 25 - 25 \times 10 = 562{\cdot}5 \text{ kg cm}$$

Shear stress due to torque,

$$q = \frac{16T}{\pi D^3} = \frac{16 \times 250}{\pi \times 2^3} = 159 \text{ kg/cm}^2$$

Direct stress due to bending,

$$f = \frac{32M}{\pi D^3} = \frac{32 \times 562{\cdot}5}{\pi \times 2^3} = 716 \text{ kg/cm}^2 \textbf{ compression}$$

Principal stresses

$$= \tfrac{1}{2} \times 716 \pm \tfrac{1}{2}\sqrt{716^2 + 4 \times 159^2}$$

$$= +749{\cdot}5 \text{ kg/cm}^2 \text{ (comp.) and } -33{\cdot}5 \text{ kg/cm}^2 \text{ (tension).}$$

14. A single-throw crank-shaft is supported in two bearings, the distance of the centre of the crank-pin being 30 cm to one of the bearings and 45 cm to the other. The crank length is 25 cm and the crank-pin is 10 cm diameter. The maximum thrust on the crank-pin is 5 tonnes and occurs when the crank and connecting rod are at right angles. Work is taken off from the crank-shaft at a point adjacent to the nearer bearing. Determine the maximum direct and shearing stresses in the crank-pin.

(Lond. Univ.)

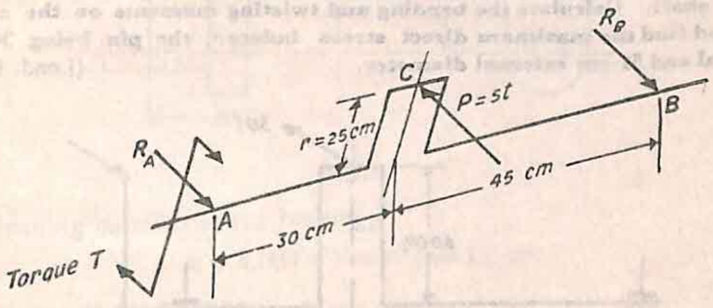

Fig. 3

Torque T transmitted through the shaft

$$= P.r = 5{,}000 \times 25 = 125{,}000 \text{ kg cm}$$

Reaction R_A at the bearing A

$$= \frac{5{,}000 \times 45}{75} = 3{,}000 \text{ kg}$$

Reaction R_B at the bearing B

$$= \frac{5,000 \times 30}{75} = 2,000 \text{ kg}$$

Bending moment on the crank-pin at C,

$$M_C = 30R_A = 45R_B = 90,000 \text{ kg cm}$$

Torque on the crank pin at C,

$$T_C = T - r.R_A = r.R_B = 50,000 \text{ kg cm}$$

Equivalent bending moment at C,

$$M_e = \tfrac{1}{2}[90,000 + \sqrt{90,000^2 + 50,000^2}]$$
$$= \tfrac{1}{2}[90,000 + 103,000] = 96,500$$

Maximum direct stress,

$$f_1 = \frac{32M_e}{\pi D^3} = \frac{32 \times 96,500}{\pi \times 10^3}$$
$$= \mathbf{983 \text{ kg/cm}^2}$$

Equivalent torque at C,

$$T_e = \sqrt{90,000^2 + 50,000^2} = 103,000$$

Maximum shear stress,

$$q_{max} = \frac{16T_e}{\pi d^3} = \frac{16 \times 103,000}{\pi \times 10^3}$$
$$= \mathbf{525 \text{ kg/cm}^2}.$$

15. The crank-shaft of a single cylinder engine is shown in Fig. 4. When the crank and connecting rod are at right angles the effective force in the rod is 30 tonnes. The work is entirely taken off at the right-hand end of the shaft, and the bearings may be assumed to exercise no restraint on the shaft. Calculate the bending and twisting moments on the crank-pin, and find the maximum direct stress induced, the pin being 30 cm internal and 52 cm external diameter. (Lond. Univ.)

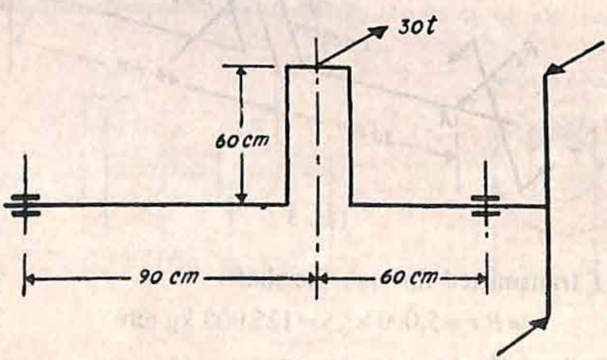

Fig. 4

(Ans. 1,080 t cm, 720 t cm; 96·9 kg/cm²)

16. (*a*) **A solid circular shaft of radius r and diametral moment of inertia I is subjected to a bending moment $M\cos\alpha$ and a twisting moment $M\sin\alpha$. Show that the maximum shearing stress has a constant value $\frac{1}{2}MK$ for all values of α where $K=\frac{r}{I}$.**

(*b*) **An engine has an overhung crank-shaft and its stroke is 30 cm. The centre line of the crank-pin and the connecting rod is 20 cm distant from the centre of the supporting bearing A thrust of 4,000 kg acts on the crank-pin at right angles to the crank. Determine the diameter of the shaft if the stresses in tension and shear are not to exceed 700 kg/cm² and 400 kg/cm² respectively.** (Engineering Services, 1959)

(a) Equivalent torque,

$$T_e=\sqrt{(M\cos\alpha)^2+(M\sin\alpha)^2}=M$$

Maximum shearing stress,

$$q_{max}=\frac{T_e}{J}\times r=\frac{M}{2I}.r$$

$$=\tfrac{1}{2}MK$$

(b) See Fig. 5.

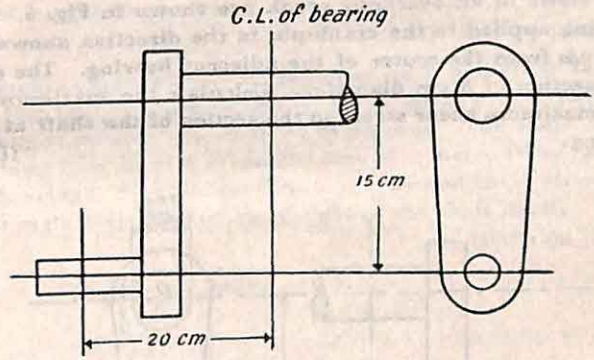

Fig. 5

Bending moment at the bearing,

$$M=4{,}000\times 20=80{,}000 \text{ kg cm}$$

Torque at the bearing,

$$T=4{,}000\times 15=60{,}000 \text{ kg cm}$$

$$M_e=\tfrac{1}{2}[80{,}000+\sqrt{80{,}000^2+60{,}000^2}]$$

$$=90{,}000 \text{ kg cm}$$

But $\qquad M_e=\frac{\pi}{32}D^3\times 700$

$$\therefore \quad \frac{\pi}{32} D^3 \times 700 = 90{,}000$$

or $$D^3 = \frac{90{,}000 \times 32}{\pi \times 700}$$

Solving $$D = 10{\cdot}94 \text{ cm}$$

$$T_e = \sqrt{80{,}000^2 + 60{,}000^2}$$
$$= 100{,}000 \text{ kg cm}$$

But $$T_e = \frac{\pi}{16} D^3 \times 400$$

$$\therefore \quad \frac{\pi}{16} D^3 \times 400 = 100{,}000$$

or $$D^3 = \frac{100{,}000 \times 16}{\pi \times 400}$$

Solving $$D = 10{\cdot}84 \text{ cm}$$

Adopt 10·94 cm diameter.

17. Two views of an overhung crank are shown in Fig. 6, a force of 2,000 kg being applied to the crank-pin in the direction shown and at a distance 16 cm from the centre of the adjacent bearing. The crank-shaft is of solid section of 8 cm diameter. Calculate the maximum principal stress and maximum shear stress in the section of the shaft at the centre of the bearing.

(Lond. Univ.)

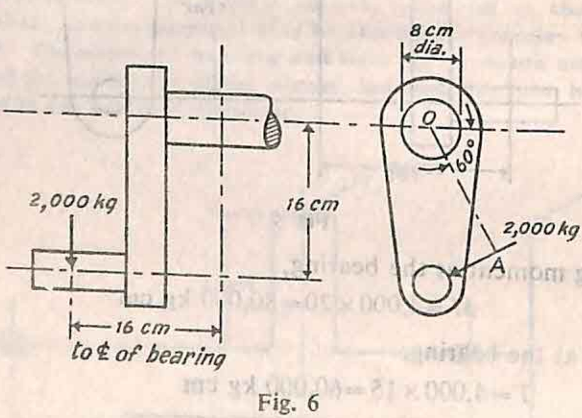

Fig. 6

At the bearing,

$$M = 2{,}000 \times 16 = 32{,}000 \text{ kg cm}$$

$$T = 2{,}000 \times \text{length } OA = 2{,}000 \times 16 \sin 60$$
$$= 27{,}700 \text{ kg cm}$$

$$M_e = \tfrac{1}{2}[32{,}000 + \sqrt{32{,}000^2 + 27{,}700^2}]$$
$$= 37{,}150 \text{ kg cm}$$

Maximum principal stress,

$$f = \frac{32M_e}{\pi D^3} = \frac{32 \times 37{,}150}{\pi \times 8^3} = 739 \text{ kg/cm}^2$$

$$T_e = \sqrt{32{,}000^2 + 27{,}700^2} = 42{,}300 \text{ kg cm}$$

Maximum shear stress,

$$q_{max} = \frac{16T_e}{\pi D^3} = \frac{16 \times 42{,}300}{\pi \times 8^3} = 421 \text{ kg/cm}^2.$$

18. A steel shaft *ABCD* of circular section is 144 cm long and is supported in bearings at the ends *A* and *D*. *AB*=54 cm, *BC*=36 cm and *CD*=54 cm. The shaft is horizontal and two horizontal arms, rigidly connected to the shaft at *B* and *C*, project from it at right angles on opposite sides. The arm at *B* carries a vertical load of 2,000 kg at 24 cm from the shaft axis, and the arm at *C* carries a vertical balancing load at 30 cm from the axis.

If the shearing stress is not to exceed 800 kg/cm², determine the minimum permissible diameter of the shaft. Assume the bearings give simple point support to the shaft. (Lond. Univ.)

(Ans. 8·9 cm.)

19. An engine shaft, 15 cm in diameter and 3 metres long between centres of bearings, transmits 250 h. p. at 80 r. p. m. The maximum twisting moment exceeds the mean by 25 percent. The shaft carries a fly-wheel weighing 2,500 kg at a mean distance of 1 metre from the centre of one of the bearings. Find the maximum tensile and shear stresses induced in the shaft material. Neglect the weight of the shaft itself.

(Engineering Services, 1965)

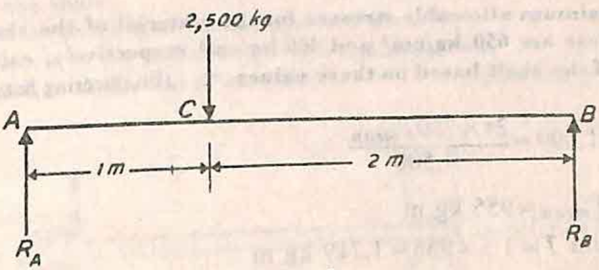

Fig. 7

$$R_A = \frac{2{,}500 \times 2}{3} = 1{,}667 \text{ kg}$$

Bending moment at C,

$$M = 1{,}667 \times 1 = 1{,}667 \text{ kg m}$$

This is the maximum bending moment.

$$\text{H. P.}=250=\frac{2\pi\times 80T_{mean}}{4,500}$$

$$T_{mean}=2,240 \text{ kg m}$$

$$\therefore \quad T_{max}=1{\cdot}25T_{mean}=2,800 \text{ kg m}$$

Equivalent bending moment,

$$M_e=\tfrac{1}{2}[1,667+\sqrt{1,667^2+2,800^2}]$$
$$=\tfrac{1}{2}[1,667+3,259]$$
$$=2,463 \text{ kg m}$$

Maximum tensile stress,

$$f_1=\frac{32M_e}{\pi D^3}=\frac{32\times 246,300}{\pi\times 15^3}=\mathbf{743\ kg/cm^2}$$

Equivalent torque,

$$T_e=\sqrt{1,667^2+2,800^2}=3,259 \text{ kg m}$$

Maximum shear stress,

$$q_{max}=\frac{16T_e}{\pi D^3}=\frac{16\times 325,900}{\pi\times 15^3}=\mathbf{492\ kg/cm^2}.$$

20. A shaft transmits 1,000 horse-power at 750 r. p. m. The maximum torque on the shaft exceeds its mean value by 80 percent. The shaft is supported in main bearings 3 metres apart and carries at its middle a flywheel weighing 5,000 kg. Due to fluid pressure on the piston the shaft is also subjected at the point where the flywheel is located to a bending moment which may be taken as numerically equal to 80 percent of the mean twisting moment.

If the maximum allowable stresses for the material of the shaft in tension and shear are 650 kg/cm² and 400 kg/cm² respectively, calculate the diameters of the shaft based on these values. (Engineering Services, 1954)

$$\text{H. P.}=1,000=\frac{2\pi\times 750T_{mean}}{4,500}$$

$$\therefore \quad T_{mean}=955 \text{ kg m}$$

Maximum $T=1{\cdot}8\times 955=1,719$ kg m

Maximum bending moment at the centre

$$=\frac{WL}{4}=\frac{5,000\times 3}{4}=3,750 \text{ kg m}$$

Bending moment due to fluid pressure

$$=0{\cdot}8\times 955=764 \text{ kg m}$$

Total $M = 3{,}750 + 764 = 4{,}514$ kg m

$$M_e = \tfrac{1}{2}[4{,}514 + \sqrt{4{,}514^2 + 1{,}719^2}] = \tfrac{1}{2}[4{,}514 + 4{,}830]$$
$$= 4{,}672 \text{ kg m} = 467{,}200 \text{ kg cm}$$

But $$M_e = \frac{\pi}{32} D^3 \times 650$$

$\therefore$ $$\frac{\pi}{32} D^3 \times 650 = 467{,}200$$

or $$D^3 = \frac{467{,}200 \times 32}{\pi \times 650}$$

Solving $$D = 19 \cdot 42 \text{ cm}$$

$$T_e = \sqrt{4{,}514^2 + 1{,}719^2} = 4{,}830 \text{kg m}$$
$$= 483{,}000 \text{ kg cm}$$

But $$T_e = \frac{\pi}{16} D^3 \times 400$$

$$\therefore \frac{\pi}{16} D^3 \times 400 = 483{,}000$$

or $$D^3 = \frac{483{,}000 \times 16}{\pi \times 400}$$

Solving $$D = 18 \cdot 32 \text{ cm}$$

Adopt **19·42 cm** diameter.

21. A propeller of 6 tonnes weight is carried by a shaft of 22 cm diameter and overhangs the supporting bracket by 44 cm. The propeller receives 4,000 h. p. at a speed of 300 r. p. m. If the propeller thrust is 15 tonnes, calculate the principal stresses at the following points on the surface of the shaft:

(*a*) **when the point is at the bottom of the shaft,**
(*b*) **when it is at the end of the horizontal diameter,**
(*c*) **when it is at the top of the shaft.** (Engineering Services, 1964)

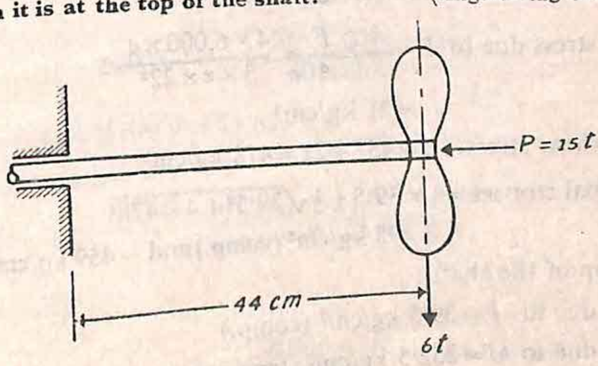

Fig. 8

At the bearing,

Shearing force, $F=6$ t

Bending moment, $M=6\times 44=264$ t cm

$$\text{H. P.}=4{,}000=\frac{2\pi\times 300T}{4{,}500}$$

$$\therefore\quad T=9{,}550 \text{ kg m}$$

(a) Bottom of the shaft :

Stress due to thrust P,

$$=\frac{P}{A}=\frac{15{,}000\times 4}{\pi\times 22^2}=39\cdot 5 \text{ kg/cm}^2 \text{ (comp.)}$$

Stress due to $M=\dfrac{32M}{\pi D^3}=\dfrac{32\times 264{,}000}{\pi\times 22^3}$

$$=252\cdot 5 \text{ kg/cm}^2 \text{ (comp.)}$$

Total direct stress,

$$f=39\cdot 5+252\cdot 5=292 \text{ kg/cm}^2 \text{ (comp.)}$$

Shear stress due to T,

$$q=\frac{16T}{\pi D^3}=\frac{16\times 955{,}000}{\pi\times 22^3}=457 \text{ kg/cm}^2$$

Shear stress due to $F=0$.

Principal stresses $=\frac{1}{2}\times 292\pm\frac{1}{2}\sqrt{292^2+4\times 457^2}$

$= +$**626 kg/cm² (comp.)** and $-$ **334 kg/cm² (tension**

(b) End of the horizontal diameter :

Stress due to P, $f=39\cdot 5$ kg/cm² (comp.)

Stress due to $M=0$

Shear stress due to $T=457$ kg/cm²

Shear stress due to $F=\dfrac{4}{3}\times\dfrac{F}{A}=\dfrac{4\times 6{,}000\times 4}{3\times\pi\times 22^2}$

$$=21 \text{ kg/cm}^2$$

Total shear stress, $q=457+21=478$ kg/cm²

Principal stresses $=\frac{1}{2}\times 39\cdot 5\pm\frac{1}{2}\sqrt{39\cdot 5^2+4\times 478^2}$

$= +$**498 kg/cm² (comp.)** and $-$ **459 kg/cm² (tension)**

(c) Top of the shaft :

Stress due to $P=39\cdot 5$ kg/cm² (comp.)

Stress due to $M=252\cdot 5$ kg/cm² (tension)

Total direct stress, $f = 252 \cdot 5 - 39 \cdot 5 = 213$ kg/cm² (tension)

Shear stress due to T, $q = 457$ kg/cm²

Shear stress due to $F = 0$

Principal stresses $= \frac{1}{2} \times 213 \pm \frac{1}{2}\sqrt{213^2 + 4 \times 457^2}$

$= +575 \cdot 5$ kg/cm² (tension) and $-362 \cdot 5$ kg/cm² (comp.)

22. A hollow shaft of external diameter 32 cm and internal diameter 20 cm, turning at 150 r. p. m., is required to drive a screw propeller fitted to a vessel whose estimated speed is 30 km/h for an expenditure of 12,000 shaft h. p., the efficiency of the propeller being assessed at 70%. Calculate the values of the principal stresses and the maximum shearing stress at a point on the outer surface.

$$A = \frac{\pi}{4}(32^2 - 20^2) = 490 \text{ cm}^2$$

$$J = \frac{\pi}{32}(32^4 - 20^4) = 87{,}200 \text{ cm}^4$$

$$\text{Shaft h. p.} = 12{,}000 = \frac{2\pi \times 150T}{4{,}500}$$

$$T = 57{,}300 \text{ kg m}$$

Work done by shaft $= 12{,}000 \times 75$ kg m/sec

Work done by propeller $= 0 \cdot 70 \times 12{,}000 \times 75$

$= 630{,}000$ kg m/sec

But work done by propeller/sec

= Thrust × distance moved/sec

$$\therefore \quad 630{,}000 = P \times \frac{30 \times 1{,}000}{60 \times 60}$$

or End thrust $P = 75{,}600$ kg

Direct stress due to P,

$$f = \frac{75{,}600}{490} = 154 \text{ kg/cm}^2$$

Shear stress due to T,

$$q = \frac{5{,}730{,}000 \times 16}{87{,}200} = 1{,}051 \text{ kg/cm}^2$$

Principal stresses $= \frac{1}{2}\times 154 \pm \frac{1}{2}\sqrt{154^2+4\times 1{,}051^2}$

$= +1{,}131$ kg/cm² and -977 kg/cm²

Maximum shear stress,

$$q_{max}=\tfrac{1}{2}(f_1-f_2)=\tfrac{1}{2}(1{,}131+977)$$
$$=1{,}054 \text{ kg/cm}^2.$$

23. A propeller shaft of a ship is 45 cm diameter and it supports a propeller of weight 15 t. The propeller can be considered as a load concentrated at the end of a cantilever of length 2 m. The propeller is driven at 100 rev./min. when the speed of the ship is 32 km/h. If the engine develops 20,000 h. p., calculate the principal stresses in the shaft and the maximum shear stress. It may be assumed that the propulsive efficiency of the propeller is 85 percent.

(Ans. $+1{,}040$ and -615 kg/cm², 827·5 kg/cm².)

24. A shaft 10 cm diameter and 2 metres long is fixed at both ends and supports a load of 2,500 kg from the rim of a pulley 60 cm in diameter as shown in Fig. 9. The weight of the pulley is 500 kg. Find the maximum bending and shearing stresses in the shaft and the maximum principal stress.

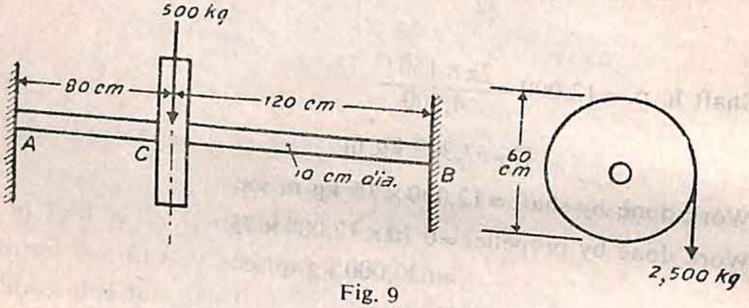

Fig. 9

Maximum bending moment at the support A,

$$M_A=\frac{(2{,}500+500)80\times 120^2}{200^2}=86{,}400 \text{ kg cm}$$

Torque transmitted, $T=2{,}500\times 30$

$=75{,}000$ kg cm

Let T_1 be the torque carried by the length AC and T_2 by the length BC.

Since twist at C will be same for the two lengths, we get

$$\frac{T_1\times 80}{CJ}=\frac{T_2\times 120}{CJ}$$

$$\therefore \quad T_1=1{\cdot}5\,T_2$$

Also $T_1+T_2=75{,}000$ kg cm

$\therefore \quad T_1=45{,}000$ kg cm, $T_2=30{,}000$ kg cm

The cross-section at A is the weakest.

Stress due to M_A, $f=\dfrac{32\,M_A}{\pi D^3}=\dfrac{32\times 86{,}400}{\pi\times 10^3}=$ **880 kg/cm²**

Shearing stress due to T_1,

$$q=\frac{16T_1}{\pi D^3}=\frac{16\times 45{,}000}{\pi\times 10^3}=\mathbf{229\ kg/cm^2}$$

Maximum principal stress,

$$f_1=\tfrac{1}{2}\times 880+\tfrac{1}{2}\sqrt{880^2+4\times 229^2}$$
$$=\mathbf{936\ kg/cm^2}.$$

25. A 6 cm diameter shaft supported in bearings carries an 80 cm diameter pulley weighing 250 kg at an overhanging end of the shaft as shown in Fig. 10. Calculate the principal tensile stress at the section A if the horizontal belt tensions are as shown.

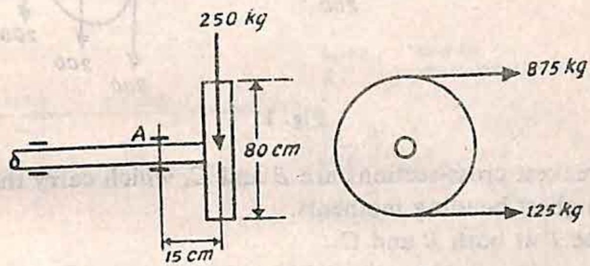

Fig. 10

At the section A,

Torque $\quad T=(875-125)40=30{,}000$ kg cm

Bending moment in the vertical plane,

$$M_1=250\times 15=3{,}750 \text{ kg cm}$$ Acc. no - 15471

Bending moment in the horizontal plane,

$$M_2=(875+125)15=15{,}000 \text{ kg cm}$$

Resultant bending moment,

$$M=\sqrt{M_1^2+M_2^2}=\sqrt{3{,}750^2+15{,}000^2}$$
$$=15{,}460 \text{ kg cm}$$

Equivalent bending moment,

$$M_e=\tfrac{1}{2}[15{,}460+\sqrt{15{,}460^2+30{,}000^2}]$$
$$=24{,}600 \text{ kg cm}$$

$$\text{Principal stress} = \frac{32M_e}{\pi D^3} = \frac{32 \times 24{,}600}{\pi \times 6^3}$$

$$= 1{,}160 \text{ kg/cm}^2.$$

25. Determine the necessary diameter for a uniform shaft carrying two equal pulleys 80 cm in diameter and weighing 200 kg each as shown in Fig. 11. The horizontal forces in the belt for one pulley and the vertical forces for the other are shown in the figure. The permissible shear stress in the shaft material is 400 kg/cm².

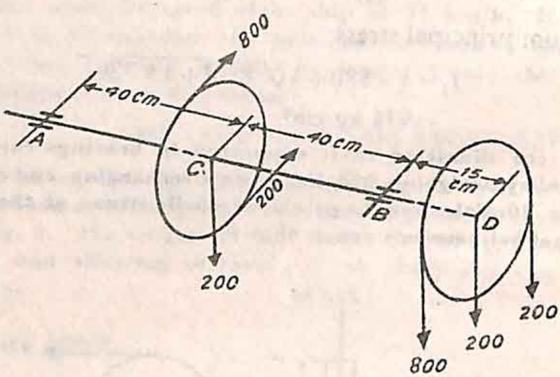

Fig. 11

The weakest cross-sections are B and C, which carry the full torque and the highest bending moments.

Torque T at both B and C

$$= (800 - 200)40 = 24{,}000 \text{ kg cm}$$

Bending moment at B in the vertical plane

$$= -(800 + 200 + 200)15 = -18{,}000 \text{ kg cm}$$

Bending moment at C in the horizontal plane

$$= \frac{(800 + 200)80}{4} = 20{,}000 \text{ kg cm}$$

Bending moment at C in the vertical plane

$$= \frac{200 \times 80}{4} - \frac{1{,}200 \times 15 \times 40}{80}$$

$$= -5{,}000 \text{ kg cm}$$

Combined bending moment at C,

$$M = \sqrt{20{,}000^2 + 5{,}000^2} = 20{,}600 \text{ kg cm}$$

This is larger than the bending moment at B.

Equivalent torque at C,

$$T_e = \sqrt{M^2+T^2} = \sqrt{20{,}600^2+24{,}000^2}$$

$$=31{,}600 \text{ kg cm}$$

But $\quad T_e = \frac{\pi}{16} D^3 \times 400$

$$\frac{\pi}{16} D^3 \times 400 = 31{,}600$$

or $\quad D^3 = \frac{31{,}600 \times 16}{\pi \times 400}$

Solving $\quad D = 7{\cdot}38$ **cm.**

27. A shaft supports the belt pulley A and the pinion B at the overhanging ends as shown in Fig. 12. The forces acting and necessary dimensions are also shown in the figure. The belt pulley is 90 cm in diameter and 160 kg in weight. The weight of the pinion is 40 kg and its pitch circle diameter is 30 cm. Calculate the pressure P on the tooth of the pinion and the diameter of the shaft if the shear stress is not to exceed 500 kg/cm².

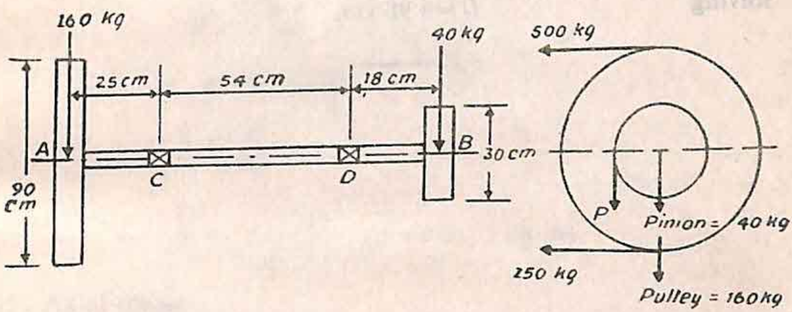

Fig. 12

For rotational equilibrium

$$P \times 15 = (750-500)45$$

$$\therefore \quad P = 750 \text{ kg}$$

The weakest cross-sections are C and D, which carry the full torque and the highest bending moments.

Torque T at both C and D

$$=(750-500)45 = 11{,}250 \text{ kg cm}$$

Bending moment at D in the vertical plane

$$=(750+40)18 = 14{,}220 \text{ kg cm}$$

Bending moment at C in the vertical plane

$$=160 \times 25 = 4{,}000 \text{ kg cm}$$

Bending moment at C in the horizontal plane

$$=(750+500)25=31{,}250 \text{ kg cm}$$

Combined bending moment at C,

$$M=\sqrt{4{,}000^2+31{,}250^2}=31{,}500 \text{ kg cm}$$

This is larger than the bending moment at D.

Equivalent torque at C,

$$T_e=\sqrt{M^2+T^2}=\sqrt{31{,}500^2+11{,}250^2}$$
$$=33{,}400 \text{ kg cm}$$

But $\quad T_e=\frac{\pi}{16}D^3\times 500$

$$\therefore \quad \frac{\pi}{16}D^3\times 500=33{,}400$$

or $\quad D^3=\frac{33{,}400\times 16}{\pi\times 500}$

Solving $\quad D=6{\cdot}98 \text{ cm.}$

CHAPTER II

CLOSE-COILED AND OPEN-COILED HELICAL SPRINGS

1. A close-coiled helical spring made of 12 mm round steel has 12 coils and the mean diameter of the coils is 16 cm. The spring is subjected to an axial load of 15 kg. Determine the elongation, intensity of torsional stress and strain energy per cubic cm under the loaded condition. $C=0\cdot84\times10^6$ kg/cm².

If the axial load is removed and an axial torque of 100 kg cm is applied, determine the axial twist, intensity of bending stress, and work stored per cubic cm in the spring. $E=2\cdot1\times10^6$ kg/cm². (Engineering Services, 1960)

(a) Axial load :

$$\delta=\frac{64WR^3n}{Cd^4}=\frac{64\times15\times8^3\times12}{0\cdot84\times10^5\times1\cdot2^4}$$

$$=3\cdot39 \text{ cm}$$

$$q=\frac{16WR}{\pi d^3}=\frac{16\times15\times8}{\pi\times1\cdot2^3}$$

$$=354 \text{ kg/cm}^2$$

$$U=\frac{q^2}{4C}.V$$

$$\therefore \quad \frac{U}{V}=\frac{354^2}{4\times0\cdot84\times10^6}=0\cdot0373 \text{ kg cm.}$$

(b) Axial torque :

$$\varphi=\frac{128MRn}{Ed^4}=\frac{128\times100\times8\times12}{2\cdot1\times10^6\times1\cdot2^4}$$

$$=0\cdot282 \text{ radian}$$

$$f=\frac{32M}{\pi d^3}=\frac{32\times100}{\pi\times1\cdot2^3}=589 \text{ kg/cm}^2$$

$$U=\frac{f^2}{8E}.V$$

$$\therefore \quad \frac{U}{V}=\frac{589^2}{8\times2\cdot1\times10^6}=0\cdot0207 \text{ kg cm.}$$

2. A closely coiled helical spring is made out of round steel wire 6 mm in diameter, the coils having a mean diameter of 8 cm. What axial pull

will produce a shear stress of 1,400 kg/cm² ? If the modulus of rigidity of the wire is 8×10^5 kg/cm² and the spring has 20 coils, how much will the spring extend under this pull and how many kg cm of work must be done in producing this extension ? (Engineering Services, 1967)

$$T = WR = \frac{\pi d^3}{16} \times q$$

or
$$W \times 4 = \frac{\pi \times (0{\cdot}6)^3 \times 1{,}400}{16}$$

$$\therefore \quad W = \mathbf{14{\cdot}84\ kg}$$

$$\delta = \frac{64 \times 14{\cdot}84 \times 4^3 \times 20}{8 \times 10^5 \times (0{\cdot}6)^4}$$

$$= \mathbf{11{\cdot}73\ cm}$$

$$\text{Work done} = \tfrac{1}{2} W\delta = \tfrac{1}{2} \times 14{\cdot}84 \times 11{\cdot}73$$

$$= \mathbf{87\ kg\ cm.}$$

3. It is required to design a close-coiled helical spring which shall deflect 1 cm under an axial load of 10 kg with a shear stress of 900 kg/cm². The spring is to be made out of round wire having a modulus of rigidity of 8×10^5 kg/cm², and the mean diameter of the coils is to be 10 times the diameter of the wire. Find the diameter and length of the wire necessary to form the spring. (Engineering Services, 1966)

$$T = WR = \frac{\pi}{16} d^3 \times q$$

or
$$10 \times 5d = \frac{\pi}{16} d^3 \times 900$$

or
$$d^2 = \frac{8}{9\pi}$$

$$\therefore \quad d = \mathbf{0{\cdot}532\ cm}$$

$$D = 10d = 5{\cdot}32 \text{ cm}$$

$$\delta = \frac{64 W R^3 n}{C d^4}$$

i. e.,
$$1 = \frac{64 \times 10 \times (5d)^3 \times n}{8 \times 10^5 \times d^4}$$

$$\therefore \quad n = 10d = 5{\cdot}32 \text{ cm}$$

$$L = \pi D n = \pi \times 5{\cdot}32 \times 5{\cdot}32$$

$$= \mathbf{88{\cdot}9\ cm.}$$

4. A close-coiled helical spring is required to have an axial stiffness of 5 kg per cm and an angular stiffness of 1 kg cm per degree angle of twist If the spring is made of steel wire 6 mm diameter, find the mean diameter of the coil and the number of turns required. Assume $E=2\times10^6$ kg/cm² and $C=0{\cdot}8\times10^6$ kg/cm². (Engineering Services, 1957)

Axial stiffness $$=\frac{W}{\delta}=\frac{Cd^4}{64R^3n}$$

i. e., $$5=\frac{0{\cdot}8\times10^6\times(0{\cdot}6)^4}{64R^3n}$$

$$\therefore \quad R^3n=324 \quad \ldots \quad (1)$$

Angular stiffness $$=\frac{M}{\varphi}=\frac{Ed^4}{128Rn}\times\frac{\pi}{180}$$

i. e., $$1=\frac{2\times10^6\times(0{\cdot}6)^4}{128Rn}\times\frac{\pi}{180}$$

$$\therefore \quad Rn=35{\cdot}3 \quad \ldots \quad (2)$$

Dividing (1) by (2),

$$R^2=\frac{324}{35{\cdot}3}=9{\cdot}18$$

$$\therefore \quad R=3{\cdot}03 \text{ cm}$$

$$\mathbf{D=6{\cdot}06 \text{ cm}}$$

From equation 2, $$n=\frac{35{\cdot}3}{3{\cdot}03}=\mathbf{11{\cdot}65.}$$

5. A close-coiled helical spring of circular section has a mean coil diameter of 8 cm. When subjected to a torque of 40 kg cm about the axis of the spring there is an angular rotation of 60 degrees, and when an axial load of 20 kg is applied, the spring extends 10·6 cm. Determine the value of Poisson's ratio for the spring material. (Lond. Univ.)

$$\varphi=\frac{128MRn}{Ed^4}$$

i. e., $$\frac{\pi}{3}=\frac{128\times40\times4n}{Ed^4}$$

$$\therefore \quad \frac{Ed^4}{n}=\frac{128\times480}{\pi} \quad \ldots \quad (1)$$

$$\delta=\frac{64WR^3n}{Cd^4}$$

i. e., $$10{\cdot}6=\frac{64\times 20\times 64n}{Cd^4}$$

$$\therefore\quad \frac{Cd^4}{n}=\frac{64\times 20\times 64}{10{\cdot}6} \qquad \ldots \qquad (2)$$

Dividing (1) by (2),

$$\frac{E}{C}=\frac{128\times 480\times 10{\cdot}6}{\pi\times 64\times 20\times 64}=2{\cdot}53$$

We know that $$E=2C\left(1+\frac{1}{m}\right)$$

or $$\frac{E}{C}=2\left(1+\frac{1}{m}\right)=2{\cdot}53$$

$$\therefore\quad \frac{1}{m}=\mathbf{0{\cdot}265.}$$

6. A close-coiled helical spring of circular section extends 1 cm when subjected to an axial load W, and there is an angular rotation of 1 radian when a torque T is independently applied about the axis. If D is the mean coil diameter, show that $\frac{T}{W}=\frac{D^2}{4}\left(1+\frac{1}{m}\right)$ where $\frac{1}{m}$ is Poisson's ratio.

Determine Poisson's ratio if $D=8$ cm, a load of 25 kg extends the spring 15 cm and a torque of 35 kg cm produces an angular rotation of 60°.

(Lond. Univ.) (Ans. 0·253.)

7. Close-coiled helical springs having n turns are made of round wire such that the mean diameter of the coils D cm is ten times the wire diameter. Show that the stiffness in kg/cm for any such spring is $\left(\frac{D}{n}\right)\times$ constant, and determine the constant when $C=8\times 10^5$ kg/cm².

Such a spring is required to support a load of 100 kg with an extension of 10 cm and a maximum shear stress of 3,500 kg/cm². Calculate (a) its weight, (b) mean coil diameter, (c) number of turns. The material weighs 0·0078 kg/cm³. (Lond. Univ.)

Stiffness $$K=\frac{W}{\delta}=\frac{Cd^4}{8D^3n}=\frac{CD^4}{8D^3n\times 10^4}$$

$$=\frac{CD}{8\times 10^4 n}$$

$$=\frac{8\times 10^5 D}{8\times 10^4 n}$$

$$=\frac{D}{n}\times 10$$

$\therefore$ Required constant = **10**

Energy stored $\quad U = \frac{1}{2}W\delta = \frac{1}{2} \times 100 \times 10$

$$= 500 \text{ kg cm}$$

Also $\quad U = \frac{q^2}{4C} \times V$

i.e., $\quad 500 = \frac{3{,}500^2}{4 \times 8 \times 10^5} \times V$

$$\therefore \quad V = \frac{6{,}400}{49} \text{ cm}^3$$

$$\text{Weight} = \frac{6{,}400}{49} \times 0{\cdot}0078 = \mathbf{1{\cdot}019 \text{ kg}}$$

$$T = \frac{WD}{2} = \frac{\pi}{16} d^3 \times q$$

or $\quad \frac{100 \times D}{2} = \frac{\pi}{16} \times \frac{D^3}{1{,}000} \times 3{,}500$

or $\quad D^2 = \frac{1{,}600}{7\pi}$

$$\therefore \quad D = \mathbf{8{\cdot}53 \text{ cm}}$$

Stiffness $\quad K = \frac{W}{\delta} = \frac{100}{10} = 10$

i.e., $\quad \frac{D}{n} \times 10 = 10$

$$\therefore \quad n = D = 8{\cdot}53.$$

8. A close-coiled helical spring is to have a stiffness of 90 kg per metre of compression, a maximum load of 4 kg and a maximum shearing stress of 1,200 kg/cm². The solid length of the spring (i.e., when the coils are touching) is to be 4·6 cm. Find the diameter of the wire, the mean radius of the coils and the number of coils required. $C = 4 \times 10^5$ kg/cm².

(Lond. Univ.)

$$T = WR = \frac{\pi}{16} d^3 \times q$$

or $\quad 4 \times R = \frac{\pi}{16} d^3 \times 1{,}200$

$$\therefore \quad R = 58{\cdot}9 d^3 \qquad (1)$$

Solid length $\quad = nd = 4{\cdot}6$

$$\therefore \quad n = \frac{4 \cdot 6}{d} \qquad \ldots \qquad (2)$$

Stiffness $$K = \frac{Cd^4}{64R^3n}$$

i.e., $$0 \cdot 9 = \frac{4 \times 10^5 d^4}{64R^3n}$$

or $$R^3n = \frac{62{,}500}{9} d^4 \qquad \ldots \qquad (3)$$

Substituting the values of R and n in equation 3,

$$(58 \cdot 9d^3)^3 \times \frac{4 \cdot 6}{d} = \frac{62{,}500}{9} d^4$$

or $$d^4 = \frac{62{,}500}{9 \times 4 \cdot 6 \times (58 \cdot 9)^3}$$

$$\therefore \quad d = \mathbf{0 \cdot 293 \ cm}$$

From equation 1, $R = 58 \cdot 9 \times (0 \cdot 293)^3 = \mathbf{1 \cdot 482 \ cm}$

From equation 2, $n = \dfrac{4 \cdot 6}{0 \cdot 293} = \mathbf{15 \cdot 7}$.

9. A close-coiled helical spring is made of steel wire 6 mm diameter coiled into 50 coils of mean diameter 5 cm. The modulus of rigidity of the steel is 8×10^5 kg/cm². The spring is held fixed at the top and a load of 15 kg is allowed to fall through a height of 5 cm before engaging with a hook at the lower end of the spring.

Calculate (*a*) the maximum extension of the spring and (*b*) the maximum stress in the wire. (Lond. Univ.)

Let δ be the maximum instantaneous extension.

Let P be the equivalent gradually applied load which would produce the same extension δ.

$$\delta = \frac{64PR^3n}{Cd^4} = \frac{64P \times 2 \cdot 5^3 \times 50}{8 \times 10^5 \times (0 \cdot 6)^4}$$

$$\therefore \quad P = 2 \cdot 07\delta$$

Loss of potential energy of weight

= Gain of strain energy of spring

i.e., $$15(5+\delta) = \tfrac{1}{2}P\delta$$

or $$15(5+\delta) = \tfrac{1}{2} \times 2 \cdot 07\delta^2$$

or $$2 \cdot 07\delta^2 - 30\delta - 150 = 0$$

Solving $\delta = \mathbf{18{\cdot}4\ cm}$

$$P = 2{\cdot}07 \times 18{\cdot}4 = 38{\cdot}1 \text{ kg}$$

Maximum stress, $$q = \frac{16PR}{\pi d^3} = \frac{16 \times 38{\cdot}1 \times 2{\cdot}5}{\pi \times (0{\cdot}6)^3}$$

$$= \mathbf{2{,}246\ kg/cm^2}.$$

10. A close-coiled helical spring is made of steel wire 6 mm diameter; it has 8 turns of mean diameter of 6·5 cm and the pitch of the coils when the spring is unloaded is 1·5 cm. Determine the axial load which when gradually applied will cause the coils just to close up. Take $C = 8 \times 10^5$ kg/cm².

If the spring stands on a rigid horizontal surface with the axis vertical and a load of 15 kg is allowed to fall freely on to the top of the coil, find the height through which it must fall in order just to close the coils together. (Lond. Univ.)

For 1 turn, $\delta = 1{\cdot}5 - 0{\cdot}6 = 0{\cdot}9$ cm

$$\delta = \frac{64WR^3n}{Cd^4}$$

i.e., $$0{\cdot}9 = \frac{64 \times W \times 6{\cdot}5^3 \times 1}{8 \times 10^5 \times (0{\cdot}6)^4 \times 8}$$

$$\therefore \quad W = \mathbf{42{\cdot}5\ kg}$$

Total displacement, $\triangle = 8 \times 0{\cdot}9 = 7{\cdot}2$ cm

Strain energy, $$U = \tfrac{1}{2}W\triangle = \tfrac{1}{2} \times 42{\cdot}5 \times 7{\cdot}2$$

$$= 153 \text{ kg cm}$$

If the 15 kg load is initially h cm above the top of the spring, then potential energy given up by this load is $15(h + 7{\cdot}2)$ kg cm.

i.e., $$15(h + 7{\cdot}2) = 153$$

$$\therefore \quad h = \mathbf{3\ cm}.$$

11. Calculate the weight of a helical spring made of steel wire of circular section which is required to bring to rest a mass of 20 kg moving with a velocity of 4 metres per second. The maximum allowable stress in the spring is to be 6,000 kg/cm² and the modulus of rigidity is 8×10^5 kg/cm². Density of steel = 0·0078 kg/cm³.

If the spring is made from wire 8 mm diameter with a mean coil diameter of 5 cm, calculate the maximum deflection produced and the number of coils required. (Lond. Univ.)

Kinetic energy of the mass $$= \frac{Wv^2}{2g} = \frac{20 \times 400^2}{2 \times 981}$$

$$= 1{,}631 \text{ kg cm}$$

Strain energy stored in the spring,

$$U=\frac{q^2}{4C}.V$$

i. e.,
$$\frac{6{,}000^2}{4\times8\times10^5}.V=1{,}631$$

$$\therefore \quad V=145 \text{ cm}^3$$

Weight of the spring $=145\times0{\cdot}0078=\mathbf{1{\cdot}131}$ **kg**

$$V=\pi Dn\times\frac{\pi}{4}d^2$$

i. e.,
$$\pi\times5\times n\times\frac{\pi}{4}\times(0{\cdot}8)^2=145$$

$$\therefore \quad n=\mathbf{18{\cdot}36}$$

Let P be the equivalent gradually applied load which would cause the same stress of 6,000 kg/cm² as is caused by the 20 kg mass moving with a velocity of 4 m/sec.

$$T=PR=\frac{\pi}{16}d^3\times q$$

or
$$P\times2{\cdot}5=\frac{\pi}{16}\times(0{\cdot}8)^3\times6{,}000$$

$$\therefore \quad P=241 \text{ kg}$$

$$U=\tfrac{1}{2}P\delta=1{,}631$$

i. e.,
$$\tfrac{1}{2}\times241\times\delta=1{,}631$$

$$\therefore \quad \delta=\mathbf{13{\cdot}54\ cm.}$$

12. Find the weight of a close-coiled helical spring which would absorb the energy of a truck weighing 10 tonnes and moving at 1 metre per second if

(*a*) the spring is compressed by the impact,

(*b*) the spring is wound up by the impact.

Working stresses: 3,500 kg/cm² in bending and 2,800 kg/cm² in torsion.

$E=2\times10^6$ **kg/cm²,** $C=0{\cdot}8\times10^6$ **kg/cm².**

The material of the spring weighs 0·0078 kg/cm³.

(Ans. 162 kg, 519 kg)

13. A close-coiled helical spring whose free length when not compressed is 15 cm is required to absorb strain energy equal to 500 kg cm when fully compressed with the coils in contact. The maximum shearing stress is limited to 1,400 kg/cm². Assuming a mean coil diameter of 10 cm find the diameter of the steel wire required and the number of coils.

$C=8\times10^5$ **kg/cm².** (Engineering Services, 1958)

Let δ be the deflection of the spring when fully compressed.

Then $$15-\delta=nd \qquad \ldots (1)$$

Strain energy, $$U=\frac{q^2}{4C}.V$$

i. e., $$500=\frac{1{,}400^2}{4\times 8\times 10^5}.V$$

$$\therefore \quad V=\frac{40{,}000}{49}\ \text{cm}^3$$

But $$V=2\pi Rn\times\frac{\pi}{4}d^2$$

$$\therefore \quad \frac{40{,}000}{49}=2\pi\times 5\times n\times\frac{\pi}{4}d^2$$

$$\therefore \quad n=\frac{33{\cdot}1}{d^2} \qquad \ldots (2)$$

$$T=WR=\frac{\pi d^3}{16}\times q$$

or $$W\times 5=\frac{\pi d^3}{16}\times 1{,}400$$

$$\therefore \quad W=55d^3$$

Strain energy $$U=500=\tfrac{1}{2}W\delta$$
$$=\tfrac{1}{2}\times 55d^3\delta$$

$$\therefore \quad \delta=\frac{18{\cdot}18}{d^3} \qquad \ldots (3)$$

Substituting the values of n and δ in equation 1

$$15-\frac{18{\cdot}18}{d^3}=\frac{33{\cdot}1}{d^2}\times d$$

or $$d\left(15-\frac{18{\cdot}18}{d^3}\right)=33{\cdot}1$$

or $$d^3-2{\cdot}21d^2-1{\cdot}212=0$$

Solving $$d=2{\cdot}42 \text{ cm}$$

From equation 2, $$n=\frac{33{\cdot}1}{2{\cdot}42^2}=5{\cdot}65.$$

14. A composite spring has two close-coiled helical steel springs connected in series; each spring has 12 coils at a mean diameter of 3 cm. Find the diameter of the wire in one of the springs if the diameter of wire in the other spring is 3 mm and the stiffness of the composite spring is 72 kg per metre.

Estimate the greatest load that can be carried by the composite spring and the corresponding extension for a maximum shearing stress of 1,900 kg/cm². $C=8\times10^5$ kg/cm². (Lond. Univ.)

Let W be the load carried by the composite spring.

Extension of the first spring,

$$\delta_1=\frac{64W\times1{\cdot}5^3\times12}{8\times10^5\times(0{\cdot}3)^4}$$

$$=0{\cdot}4W \text{ cm}$$

Extension of the second spring,

$$\delta_2=\frac{64W\times1{\cdot}5^3\times12}{8\times10^5 d^4}$$

$$=\frac{0{\cdot}00324W}{d^4} \text{ cm}$$

$\therefore$ Total elongation of the spring,

$$\delta=\delta_1+\delta_2$$

$$=0{\cdot}4W+\frac{0{\cdot}00324W}{d^4}$$

But $$\delta=\frac{W}{K}=\frac{W}{0{\cdot}72}$$

$$\therefore\quad 0{\cdot}4W+\frac{0{\cdot}00324W}{d^4}=\frac{W}{0{\cdot}72}$$

or $$\frac{0{\cdot}00324}{d^4}=0{\cdot}989$$

or $$d^4=\frac{324}{98900}$$

Solving $$d=0{\cdot}239 \text{ cm}$$

$$T=WR=\frac{\pi d^3}{16}\cdot q$$

or $$W=\frac{\pi d^3 q}{16R}$$

The limiting load will be found in the spring with the smaller wire diameter, i. e.,

$$W=\frac{\pi\times(0{\cdot}239)^3\times 1{,}900}{16\times 1{\cdot}5}$$

$$=3{\cdot}4 \text{ kg}$$

Total extension, $\delta=\frac{W}{K}=\frac{3{\cdot}4}{0{\cdot}72}$

$$=4{\cdot}72 \text{ cm.}$$

15. Two close-coiled helical springs are compressed between two parallel plates by a load of 100 kg. The springs have a wire diameter of 2 cm and the radii of the coils are 5 cm and 7 cm. Each spring has 12 coils and is of the same initial length. If the smaller spring is placed inside the larger one such that both springs deflect the same amount, determine the deflection of the springs and the maximum stress in each.

$C=6\times10^5$ kg/cm².

Let P_1 and P_2 be the loads carried by the smaller and the larger springs.

Then $\delta=\frac{64P_1\times5^3\times12}{C\times2^4}=\frac{64P_2\times7^3\times12}{C\times2^4}$

or $P_1\times5^3=P_2\times7^3$

$\therefore\ P_1=2{\cdot}744P_2$ .. (1)

Also $P_1+P_2=100$ .. (2)

Solving equations 1 and 2

$$P_1=73{\cdot}3 \text{ kg and } P_2=26{\cdot}7 \text{ kg}$$

Hence $\delta=\frac{64\times73{\cdot}3\times5^3\times12}{6\times10^5\times2^4}=0{\cdot}733$ cm

Shear stress in the smaller spring,

$$q_1=\frac{16\times73{\cdot}3\times5}{\pi\times2^3}=233 \text{ kg/cm}^2$$

Shear stress in the larger spring,

$$q_2=\frac{16\times26{\cdot}7\times7}{\pi\times2^3}=119 \text{ kg/cm}^2.$$

16. In a compound helical spring the inner spring is arranged within and concentric with the outer one, but is 1 cm shorter. The outer spring has 10 coils of mean diameter 3 cm and the wire diameter is 3 mm. Find the stiffness of the inner spring if an axial load of 10 kg causes the outer one to compress 2 cm.

If the radial clearance between the springs is 2 mm find the wire diameter of the inner spring when it has 8 coils. $C=8\times10^5$ kg/cm².

(Lond. Univ.)

Load carried by the outer spring for a compression of 2 cm

$$=\frac{Cd^4}{64R^3n}\cdot\delta=\frac{8\times10^5\times(0\cdot3)^4\times2}{64\times(1\cdot5)^3\times10}$$

$$=6 \text{ kg}$$

Hence the load carried by the inner spring

$$=10-6=4 \text{ kg, for a compression of 1 cm.}$$

Stiffness of the inner spring, $K=4$ **kg/cm**

$$D=3-0\cdot3-2\times0\cdot2-d=2\cdot3-d$$

$$K=4=\frac{Cd^4}{8D^3n}$$

$$=\frac{8\times10^5d^4}{8\times(2\cdot3-d)^3\times8}$$

or $$d^4=\frac{(2\cdot3-d)^3}{3{,}125}$$

Since d is small compared with 2·3, for first approximation

$$d^4=\frac{2\cdot3^3}{3{,}125}\quad\therefore\quad d=0\cdot25 \text{ cm}$$

Second approximation

$$d^4=\frac{(2\cdot3-0\cdot25)^3}{3{,}125}\quad\therefore\quad d=0\cdot229 \text{ cm}$$

Third and final approximation

$$d^4=\frac{(2\cdot3-0\cdot229)^3}{3{,}125}\quad\therefore\quad d=\mathbf{0\cdot231 \text{ cm.}}$$

17. A vertical rod, 3 m long, 25 mm in diameter, fixed at the top end, is provided with a collar at the bottom end. A helical spring of mean diameter 24 cm, consisting of 5 coils of 4 cm diameter steel, is mounted on the collar. A sliding weight of 550 kg is dropped down the rod on to the spring. Find the height, measured from the top of the uncompressed spring, from which the weight should be dropped to produce an instantaneous stress of 700 kg/cm² in the rod.

Find also the maximum shearing stress in the spring.

Take E for rod as 7×10^5 kg/cm² and C for spring as 8×10^5 kg/cm².

Assume the spring close-coiled, but not quite closed up tight by the action of the falling weight.

(Lond. Univ.)

Elongation of the rod,

$$\delta_1 = \frac{700}{7\times 10^5}\times 300 = 0{\cdot}3 \text{ cm}$$

Let P be the equivalent gradually applied load which would produce the same stress of 700 kg/cm² in the rod as is caused by the falling weight.

$$P = 700\times\frac{\pi}{4}(2{\cdot}5)^2 = 3{,}440 \text{ kg}$$

This load will also act on the spring.

Deflection of the spring,

$$\delta_2 = \frac{64\times 3{,}440\times 12^3\times 5}{8\times 10^5\times 4^4} = 9{\cdot}29 \text{ cm}$$

Loss of potential energy of the weight

=Strain energy stored in the rod and the spring.

i. e., $$550(h+\delta_1+\delta_2) = \tfrac{1}{2}P\delta_1 + \tfrac{1}{2}P\delta_2$$

or $$550(h+9{\cdot}59) = \tfrac{1}{2}\times 3{,}440\times 9{\cdot}59$$

$$\therefore\quad \mathbf{h = 20{\cdot}4 \text{ cm}}$$

Shear stress in the spring,

$$q = \frac{16\times 3{,}440\times 12}{\pi\times 4^3}$$

$$= \mathbf{3{,}280 \text{ kg/cm}^2}.$$

18. Find the deflection and the angular twist of the free end of a helical spring of ten coils 25 cm diameter, made of 1 cm round steel, due to an axial load of 10 kg, if the helix makes an angle of 60° with the axis (i.e., $\alpha=30°$). Estimate also the greatest intensities of direct and shear stresses in the material. $E=20\times 10^5$ **kg/cm²,** $C=8\times 10^5$ **kg/cm².**

$$\delta = \frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{64\times 10\times 12{\cdot}5^3\times 10\times 2}{\sqrt{3}}\left(\frac{3}{4\times 8\times 10^5}+\frac{2}{4\times 20\times 10^5}\right)$$

$$= \mathbf{17{\cdot}14 \text{ cm}}$$

$$\varphi = \frac{64WR^2n\sin\alpha}{d^4}\left(\frac{1}{C}-\frac{2}{E}\right)$$

$$= \frac{64\times 10\times 12{\cdot}5^2\times 10}{2}\left(\frac{1}{8\times 10^5}-\frac{2}{20\times 10^5}\right)$$

$$=\frac{1}{8}\text{ radian}=\frac{1}{8}\times\frac{180}{\pi}\text{ degrees}$$

$$=7{\cdot}16\text{ degrees}$$

Torque, $T=WR\cos\alpha$

Bending moment, $B=WR\sin\alpha$

Equivalent bending moment,

$$B_e=\tfrac{1}{2}[B+\sqrt{B^2+T^2}]$$

$$=\tfrac{1}{2}[WR\sin\alpha+\sqrt{W^2R^2\sin^2\alpha+W^2R^2\cos^2\alpha}]$$

$$=\frac{WR}{2}(1+\sin\alpha)$$

$$=\frac{10\times 12{\cdot}5}{2}\left(1+\frac{1}{2}\right)$$

$$=93{\cdot}75\text{ kg cm}$$

Maximum direct stress

$$f_1=\frac{32B_e}{\pi d^3}=\frac{32\times 93{\cdot}75}{\pi}$$

$$=955\text{ kg/cm}^2$$

Equivalent torque,

$$T_e=\sqrt{B^2+T^2}=\sqrt{W^2R^2\sin^2\alpha+W^2R^2\cos^2\alpha}$$

$$=WR$$

$$=10\times 12{\cdot}5=125\text{ kg cm}$$

Maximum shear stress,

$$q_{max}=\frac{16T_e}{\pi d^3}=\frac{16\times 125}{\pi}$$

$$=637\text{ kg/cm}^2.$$

19. An open-coiled helical spring having 8 turns is made of steel wire of 6 mm diameter. It has a mean coil diameter of 8 cm, and a pitch of 6·5 cm. If the spring is subjected to an axial torque of 50 kg cm, determine (*a*) the maximum direct stress in the wire, (*b*) the maximum shearing stress in the wire, (*c*) the angle of rotation of the spring.

$E=20\times 10^5$ kg/cm², $C=8\times 10^5$ kg/cm².

(Lond. Univ.)

$$\tan\alpha=\frac{p}{\pi D}=\frac{6{\cdot}5}{\pi\times 8}=0{\cdot}2586$$

$$\therefore\quad \alpha=14^\circ 30'$$

$$\sin\alpha = 0.2504, \ \cos\alpha = 0.9681$$
$$\sin^2\alpha = 0.0627, \ \cos^2\alpha = 0.9372$$

Torque, $T = M\sin\alpha$

Bending moment, $B = M\cos\alpha$

Equivalent bending moment,

$$B_e = \tfrac{1}{2}[M\cos\alpha + \sqrt{M^2\cos^2\alpha + M^2\sin^2\alpha}]$$
$$= \frac{M}{2}(1 + \cos\alpha)$$
$$= \tfrac{50}{2}(1 + 0.9681)$$
$$= 49.2 \text{ kg cm}$$

Maximum direct stress,

$$f_1 = \frac{32B_e}{\pi d^3} = \frac{32 \times 49.2}{\pi \times (0.6)^3}$$
$$= \mathbf{2{,}320 \text{ kg/cm}^2}$$

Equivalent torque,

$$T_e = \sqrt{M^2\cos^2\alpha + M^2\sin^2\alpha} = M$$
$$= 50 \text{ kg cm}$$

Maximum shearing stress,

$$q_{max} = \frac{16T_e}{\pi d^3} = \frac{16 \times 50}{\pi \times (0.6)^3}$$
$$= \mathbf{1{,}179 \text{ kg/cm}^2}$$

Angle of rotation,

$$\varphi = \frac{64MRn}{d^4\cos\alpha}\left(\frac{\sin^2\alpha}{C} + \frac{2\cos^2\alpha}{E}\right)$$
$$= \frac{64 \times 50 \times 4 \times 8}{(0.6)^4 \times 0.9681}\left(\frac{0.0627}{8 \times 10^5} + \frac{2 \times 0.9372}{20 \times 10^5}\right)$$
$$= \mathbf{0.829 \text{ radian.}}$$

20. An open-coiled helical spring of 5 cm mean diameter is made of steel of 6 mm diameter. Calculate the number of turns required in the spring to give a deflection of 1·2 cm for an axial load of 25 kg if the angle of helix is 30°. Calculate also the rotation of one end of the spring relative to the other if it is subjected to an axial couple of 100 kg cm. (Lond. Univ.)

$E = 2{\cdot}1 \times 10^6$ **kg/cm²,** $C = 0{\cdot}84 \times 10^6$ **kg/cm².**

(Ans. 4·77, 0·689 radian.)

21. The diameter of the wire of a helical spring with 6 coils is 1 cm, the mean coil diameter 8 cm, the pitch of the coils 8 cm. Calculate the torque which, acting coaxially with the spring, will produce a shearing stress of 3,000 kg/cm². Calculate also the angular twist of the spring produced by this torque. $C=8\times10^5$ kg/cm²., $E=20\times10^5$ kg/cm². (Lond. Univ.)

$$\tan\alpha=\frac{p}{\pi D}=\frac{8}{\pi\times 8}=0\cdot3183$$

$$\therefore\ \alpha=17^\circ39'$$

$$\sin\alpha=0\cdot3032,\quad \cos\alpha=0\cdot9529$$

$$\sin^2\alpha=0\cdot0919,\ \cos^2\alpha=0\cdot9080$$

Let M be the axial torque.

Then $$T=M\sin\alpha,$$
$$B=M\cos\alpha$$

Equivalent torque, $T_e=\sqrt{B^2+T^2}$

$$=\sqrt{M^2\cos^2\alpha+M^2\sin^2\alpha}=M$$

$$T_e=\frac{\pi d^3}{16}\times q_{max}$$

or $$M=\frac{\pi}{16}\times3{,}000=589 \text{ kg cm}$$

$$\varphi=\frac{64MRn}{d^4\cos\alpha}\left(\frac{\sin^2\alpha}{C}+\frac{2\cos^2\alpha}{E}\right)$$

$$=\frac{64\times589\times4\times6}{0\cdot9529}\left(\frac{0\cdot0919}{8\times10^5}+\frac{2\times0\cdot9080}{20\times10^5}\right)$$

$$=\mathbf{0\cdot971 \text{ radian.}}$$

22. Find the mean radius of an open-coiled spring of helix angle 30°, to give a vertical displacement of 2·3 cm and an angular rotation of the loaded end of 0·02 radian under an axial load of 4 kg. The material available is steel rod of 6 mm diameter. $E=20\times10^5$ kg/cm². $C=8\times10^5$ kg/cm².

(Lond. Univ.)

$$\delta=\frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)$$

i.e., $$2\cdot3=\frac{64\times4\times R^3n\times2}{(0\cdot6)^4\times\sqrt{3}}\left(\frac{3}{4\times8\times10^5}+\frac{2}{4\times20\times10^5}\right)$$

$$\therefore\ R^3n=849 \qquad \ldots (1)$$

$$\varphi=\frac{64WR^2n\sin\alpha}{d^4}\left(\frac{1}{C}-\frac{2}{E}\right)$$

i.e., $$0{\cdot}02=\frac{64\times 4\times R^2n}{(0{\cdot}6)^4\times 2}\left(\frac{1}{8\times 10^5}-\frac{2}{20\times 10^5}\right)$$

$$\therefore\quad R^2n=81 \qquad .. \qquad (2)$$

Dividing equation 1 by equation 2,

$$R=\frac{849}{81}=\mathbf{10{\cdot}48.}$$

23. In an open-coiled spring of ten coils the stresses due to bending and twisting are 1,000 kg/cm² and 1,100 kg/cm² respectively when the spring is loaded axially. Assuming the mean diameter of the coils to be eight times the wire diameter, find the maximum permissible axial load and the wire diameter for a maximum extension of 1·8 cm. $E=20\times 10^5$ kg/cm², $C=8\times 10^5$ kg/cm². (Lond. Univ.)

Bending moment, $$B=WR\sin\alpha$$

$$=\frac{\pi d^3}{32}\times f$$

$$\therefore\quad W\times 4d\sin\alpha=\frac{\pi d^3}{32}\times 1{,}000$$

$$\therefore\quad W\sin\alpha=24{\cdot}5d^2 \qquad .. \qquad (1)$$

Torque, $$T=WR\cos\alpha$$

$$=\frac{\pi d^3}{16}\times q$$

$$\therefore\quad W\times 4d\cos\alpha=\frac{\pi d^3}{16}\times 1{,}100$$

$$\therefore\quad W\cos\alpha=54d^2 \qquad .. \qquad (2)$$

Dividing equation 1 by equation 2,

$$\tan\alpha=\frac{24{\cdot}5}{54}=0{\cdot}4537$$

$$\therefore\quad \alpha=24^\circ\ 24'$$

$$\sin\alpha=0{\cdot}4131,\ \cos\alpha=0{\cdot}9107$$

$$\sin^2\alpha=0{\cdot}1707,\ \cos^2\alpha=0{\cdot}8294$$

From equation 1, $$W=\frac{24{\cdot}5d^2}{0{\cdot}4131}=59{\cdot}3d^2 \qquad (3)$$

$$\delta=\frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)$$

i.e., $$1{\cdot}8=\frac{64\times W\times 64d^3\times 10}{d^4\times 0{\cdot}9107}\left(\frac{0{\cdot}8294}{8\times 10^5}+\frac{2\times 0{\cdot}1707}{20\times 10^5}\right)$$

$$\therefore \quad W=33{\cdot}1d \qquad \ldots \quad (4)$$

From equations 3 and 4,

$$59{\cdot}3d^2=33{\cdot}1d$$

$$\therefore \quad d=0{\cdot}558 \text{ cm}$$

From equation 4, $$W=33{\cdot}1\times 0{\cdot}558$$

$$=\mathbf{18{\cdot}47\ kg.}$$

24. If the close-coiled spring formula is used in finding the extension of an open-coiled spring under axial load, determine the maximum angle of helix for which the error in the value of the extension is not to exceed 1 percent. Assume $E=2{\cdot}5\ C$. (Lond. Univ.)

Extension according to close-coiled formula,

$$\delta_1=\frac{64WR^3n}{Cd^4}$$

Extension according to open-coiled formula,

$$\delta_2=\frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)$$

Percentage error $=1=\dfrac{\delta_2-\delta_1}{\delta_2}\times 100$

$$=\frac{\dfrac{1}{\cos\alpha}\left(\dfrac{\cos^2\alpha}{C}+\dfrac{2\sin^2\alpha}{E}\right)-\dfrac{1}{C}}{\dfrac{1}{\cos\alpha}\left(\dfrac{\cos^2\alpha}{C}+\dfrac{2\sin^2\alpha}{E}\right)}\times 100$$

$$=\frac{E\cos^2\alpha+2C\sin^2\alpha-E\cos\alpha}{E\cos^2\alpha+2C\sin^2\alpha}\times 100$$

$$=\frac{2{\cdot}5C\cos^2\alpha+2C\sin^2\alpha-2{\cdot}5C\cos\alpha}{2{\cdot}5C\cos^2\alpha+2C\sin^2\alpha}\times 100$$

or $$2{\cdot}5\cos^2\alpha+2\sin^2\alpha=250\cos^2\alpha+200\sin^2\alpha-250\cos\alpha$$

or $$247{\cdot}5\cos^2\alpha+198\sin^2\alpha-250\cos\alpha=0$$

or $$49{\cdot}5\cos^2\alpha-250\cos\alpha+198=0$$

or $$\cos^2\alpha-5{\cdot}05\cos\alpha+4=0$$

Solving $$\cos\alpha = 0{\cdot}985$$

$$\therefore \quad \alpha = 9^\circ\ 54'.$$

25. Calculate the percentage error in the value obtained for the stiffness if the inclination of the coils is neglected for a helical spring in which the angle of helix $\alpha=30^\circ$. Take $E=2{\cdot}5\ C$. (Lond. Univ.)

For close-coiled spring

Stiffness, $$K_1 = \frac{W}{\delta} = \frac{Cd^4}{64R^3n}$$

For open-coiled spring

$$\delta = \frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{64WR^3n \times 2}{d^4 \times \sqrt{3}}\left(\frac{3}{4C} + \frac{2}{4 \times 2{\cdot}5C}\right)$$

$$= \frac{608WR^3n}{5\sqrt{3}Cd^4}$$

$\therefore$ Stiffness, $$K_2 = \frac{W}{\delta} = \frac{5\sqrt{3}Cd^4}{608R^3n}$$

Percentage error $$= \frac{K_2 - K_1}{K_2} \times 100 = \frac{\dfrac{5\sqrt{3}}{608} - \dfrac{1}{64}}{\dfrac{5\sqrt{3}}{608}} \times 100$$

$$= \frac{10\sqrt{3} - 19}{10\sqrt{3}} \times 100$$

$$= -9{\cdot}7.$$

26. How many springs, each of 25 coils of 20 cm external diameter and made of 2 cm round steel wire, are necessary to stop a 2,000 kg truck moving at 8 km per hour without the springs compressing more than 25 cm ? Assume springs to be open-coiled, α being 30°. $E=20\times10^5$ kg/cm² $C=8\times10^5$ kg/cm². What shear stress will be induced in the wire ? $g=981$ cm/sec².

$$v = 8 \text{ km/hour} = \frac{8\times10^5}{3{,}600} \text{ cm/sec}$$

$$= \frac{2{,}000}{9} \text{ cm/sec}$$

Kinetic energy of the truck

$$= \frac{Wv^2}{2g} = \frac{2{,}000 \times \left(\dfrac{2{,}000}{9}\right)^2}{2 \times 981}$$

$$=50{,}300 \text{ kg cm}$$

Let P be the equivalent gradually applied load which would produce the same deflection of 25 cm.

$$\delta=\frac{64PR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)$$

i.e.,
$$25=\frac{64P\times 9^3\times 25\times 2}{2^4\times\sqrt{3}}\left(\frac{3}{4\times 8\times 10^5}+\frac{2}{4\times 20\times 10^5}\right)$$

$$\therefore\quad P=250 \text{ kg}$$

Energy stored in one spring

$$=\tfrac{1}{2}P\delta=\tfrac{1}{2}\times 250\times 25=3{,}125 \text{ kg cm}$$

Number of springs required

$$=\frac{50{,}300}{3{,}125}=16{\cdot}1$$

Provide **17** number of springs.

Torque, $T=PR\cos\alpha$

Bending moment, $B=PR\sin\alpha$

Equivalent torque, $T_e=\sqrt{B^2+T^2}=PR$

$\therefore$ Maximum shear stress,

$$q_{max}=\frac{16T_e}{\pi d^3}=\frac{16PR}{\pi d^3}=\frac{16\times 250\times 9}{\pi\times 8}$$

$$=\mathbf{1{,}432 \text{ kg/cm}^2}.$$

27. An open-coiled spring of 12 cm mean diameter has 10 coils of 2 cm diameter wire, at a slope of 30° to the horizontal when the coil axis is vertical. Find the axial load and torque necessary to extend the spring 0·5 cm, if rotation is prevented, indicating whether the torque tends to wind up or unwind the spring.

$E=20\times 10^5$ kg/cm², $C=8\times 10^5$ kg/cm². (Lond Univ.)

$$\varphi=\frac{64WR^2n\sin\alpha}{d^4}\left(\frac{1}{C}-\frac{2}{E}\right)+\frac{64MRn}{d^4\cos\alpha}\left(\frac{\sin^2\alpha}{C}+\frac{2\cos^2\alpha}{E}\right)=0$$

or
$$WR\sin\alpha\left(\frac{1}{C}-\frac{2}{E}\right)+\frac{M}{\cos\alpha}\left(\frac{\sin^2\alpha}{C}+\frac{2\cos^2\alpha}{E}\right)=0$$

or $$\frac{W\times 6}{2}\left(\frac{1}{8\times 10^5}-\frac{2}{20\times 10^5}\right)+\frac{2M}{\sqrt{3}}\left(\frac{1}{4\times 8\times 10^5}+\frac{2\times 3}{4\times 20\times 10^5}\right)=0$$

or $$\frac{3W}{40\times 10^5}+\frac{17M}{80\sqrt{3}\times 10}=0$$

$$\therefore\quad M=-\frac{6\sqrt{3}}{17}W=-0{\cdot}611\ W$$

$$\delta=\frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)+\frac{64MR^2n\sin\alpha}{d^4}\left(\frac{1}{C}-\frac{2}{E}\right)$$

$$=0{\cdot}5\text{ cm}$$

or $$\frac{64\times W\times 6^3\times 10\times 2}{16\times\sqrt{3}}\left(\frac{3}{4\times 8\times 10^5}+\frac{2}{4\times 20\times 10^5}\right)+\frac{64M\times 6^2\times 10}{16\times 2}\left(\frac{1}{8\times 10^5}-\frac{2}{20\times 10^5}\right)=0{\cdot}5$$

or $$\frac{684\sqrt{3}W}{10^5}+\frac{18M}{10^5}=0{\cdot}5$$

or $$684\sqrt{3}W+18\times(-0{\cdot}611W)=0{\cdot}5\times 10^5$$

or $$1{,}174W=0{\cdot}5\times 10^5$$

$$\therefore\quad \mathbf{W=42{\cdot}6\text{ kg}}$$

$$M=-0{\cdot}611\times 42{\cdot}6=\mathbf{-26\text{ kg cm}}$$

The negative sign indicates an unwinding torque.

28. Two shafts in line, which are prevented from moving axially, are connected by a helical spring, the spring fitting loosely on the shafts and having its ends fixed to the shafts. Show that, if the coils of the spring are of circular cross-section, and are inclined at 45° to the axis, the couple per unit angle of twist is given by $\frac{d^4}{128\sqrt{2}Rn}\left(\frac{E}{2}+C\right)$ where d is the diameter of the cross-section, R is the mean radius of the coils and n is the number of coils.

$$\delta=\frac{64WR^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)+\frac{64MR^2n\sin\alpha}{d}\left(\frac{1}{C}-\frac{2}{E}\right)=0$$

or $$\frac{WR}{\cos\alpha}\left(\frac{\cos^2\alpha}{C}+\frac{2\sin^2\alpha}{E}\right)+M\sin\alpha\left(\frac{1}{C}-\frac{2}{E}\right)=0$$

or $$WR\sqrt{2}\left(\frac{1}{2C}+\frac{2}{2E}\right)+\frac{M}{\sqrt{2}}\left(\frac{1}{C}-\frac{2}{E}\right)=0$$

or $$W=-\frac{M}{R}\times\frac{E-2C}{E+2C}$$

$$\varphi=\frac{64WR^2n\sin\alpha}{d^4}\left(\frac{1}{C}-\frac{2}{E}\right)+\frac{64MRn}{d^4\cos\alpha}\left(\frac{\sin^2\alpha}{C}+\frac{2\cos^2\alpha}{E}\right)$$

$$=\frac{64WR^2n}{d^4\sqrt{2}}\left(\frac{1}{C}-\frac{2}{E}\right)+\frac{64MRn\sqrt{2}}{d^4}\left(\frac{1}{2C}+\frac{2}{2E}\right)$$

$$=\frac{32\sqrt{2}Rn}{d^4}\left[WR\left(\frac{1}{C}-\frac{2}{E}\right)+2M\left(\frac{1}{2C}+\frac{1}{E}\right)\right]$$

$$=\frac{32\sqrt{2}Rn}{d^4}\left[-M\times\frac{E-2C}{E+2C}\left(\frac{1}{C}-\frac{2}{E}\right)+2M\left(\frac{1}{2C}+\frac{1}{E}\right)\right]$$

$$=\frac{32\sqrt{2}Rn}{d^4}\left[\frac{-M(E-2C)^2}{EC(E+2C)}+\frac{M(E+2C)}{EC}\right]$$

$$=\frac{32\sqrt{2}Rn}{d^4}\times\frac{8M}{E+2C}$$

$$=\frac{256\sqrt{2}MRn}{d^4(E+2C)}$$

$$\therefore\quad \frac{M}{\varphi}=\frac{d^4(E+2C)}{256\sqrt{2}Rn}=\frac{d^4}{128\sqrt{2}Rn}\left(\frac{E}{2}+C\right)$$

CHAPTER III

LEAF, FLAT SPIRAL AND CONICAL SPRINGS

1. A steel carriage spring is to be 75 cm long and to carry a central load of 500 kg. If the plates are 8 cm wide and 5 mm thick, how many plates will be required if the stress is to be limited to 1,800 kg/cm² ? What will be the deflection of the spring at the centre ? To what radius should each piece be curved so that it straightens out under the load ?

Take $E=2\times10^6$ kg/cm².

$$f=\frac{3WL}{2nbt^2}$$

i.e.,

$$1{,}800=\frac{3\times500\times75}{2n\times8\times(0\cdot5)^2}$$

$$\therefore\quad n=15\cdot6$$

Adopt 16 plates.

$$\delta=\frac{3WL^3}{8Enbt^3}=\frac{3\times500\times75^3}{8\times2\times10^6\times16\times8\times(0\cdot5)^3}$$

$$=2\cdot47 \text{ cm}$$

$$\delta=\frac{L^2}{8R}$$

$$\therefore\quad R=\frac{75^2}{8\times2\cdot47}=285 \text{ cm.}$$

2. A laminated spring, made of 12 steel plates, is 90 cm long. The maximum central load is 800 kg. If the maximum allowable stress in the steel is 2,500 kg/cm² and the maximum deflection 3·5 cm, calculate the width and thickness of the plates. $E=2\cdot1\times10^6$ kg/cm². (Lond. Univ.)

$$f=\frac{3WL}{2nbt^2}$$

i. e.,

$$2{,}500=\frac{3\times800\times90}{2\times12\ bt^2}$$

$$\therefore\quad bt^2=3\cdot6 \tag{1}$$

$$\delta=\frac{3WL^3}{8Enbt^3}$$

i e., $$3{\cdot}5=\frac{3\times800\times90^3}{8\times2{\cdot}1\times10^6\times12bt^3}$$

$$\therefore \quad bt^3=2{\cdot}48 \qquad \ldots \qquad (2)$$

Dividing equation 2 by equation 1

$$t=\frac{2{\cdot}48}{3{\cdot}6}=0{\cdot}689 \text{ cm}$$

From equation 1, $$b=\frac{3{\cdot}6}{(0{\cdot}689)^2}=7{\cdot}58 \text{ cm.}$$

3. A carriage spring 130 cm long has leaves of 10 cm width and 12 mm thickness. The maximum bending stress is 1,500 kg/cm² and the spring must absorb 1,200 kg cm of energy when straightened. Calculate the number of leaves and their initial radius of curvature. $E=2\times10^6$ kg/cm².
(Lond. Univ.)

Energy stored, $$U=\frac{f^2}{6E}\cdot V=\frac{f^2}{6E}\cdot\frac{nbtL}{2}$$

i. e., $$1{,}200=\frac{1{,}500^2\times n\times10\times1{\cdot}2\times130}{6\times2\times10^6\times2}$$

$$\therefore \quad n=8{\cdot}21$$

Adopt 9 plates.

The actual bending stress f_1 is given by

$$1{,}200=\frac{f_1^2\times9\times10\times1{\cdot}2\times130}{6\times2\times10^6\times2}$$

$$\therefore \quad f_1=1{,}432 \text{ kg/cm}^2$$

The bending stresses in the plates when straightened will be the reversed stresses that would obtain if the plates were originally straight and then bent to the radius R.

i.e., $$\frac{2f_1}{t}=\frac{E}{R}$$

$$\therefore \quad R=\frac{Et}{2f_1}=\frac{2\times10^6\times1{\cdot}2}{2\times1{,}432}$$

$$=838 \text{ cm.}$$

4. A laminated steel spring, simply supported at the ends and centrally loaded, with a span of 75 cm is required to carry a proof load of 750 kg and the central deflection is not to exceed 5 cm. The bending stress must not be greater than 4,000 kg/cm². Plates are available in multiples of 1 mm for thickness and in multiples of 4 mm for width.

Determine suitable values for the thickness, width and number of plates and the radius to which the plates should be formed.

Assume the width to be twelve times the thickness. $E=2\times 10^6$ kg/cm².

(Lond. Univ.)

$$\delta=\frac{3WL^3}{8Enbt^3}$$

i.e ,

$$5=\frac{3\times 750\times 75^3}{8\times 2\times 10^6\times n\times 12t\times t^3}$$

$$\therefore \quad nt^4=0{\cdot}989 \qquad \ldots \qquad (1)$$

$$f=\frac{3WL}{2nbt^2}$$

i.e.,

$$4{,}000=\frac{3\times 750\times 75}{2n\times 12t\times t^2}$$

$$\therefore \quad nt^3=1{\cdot}758 \qquad \ldots \qquad (2)$$

Dividing (1) by (2),

$$t=\frac{0{\cdot}989}{1{\cdot}758}=0{\cdot}563 \text{ cm}$$

Adopt **6 mm** thick plates.

$$b=12\ t=72 \text{ mm}=\mathbf{7{\cdot}2\ cm}$$

From equation 2, $$n=\frac{1{\cdot}758}{(0{\cdot}6)^3}=8{\cdot}14$$

Adopt **9** plates.

Actual deflection under the proof load,

$$\delta_1=\frac{3\times 750\times 75^3}{8\times 2\times 10^6\times 9\times 7{\cdot}2\times(0{\cdot}6)^3}$$

$$=4{\cdot}24 \text{ cm}$$

Initial radius of curvature,

$$R=\frac{L^2}{8\delta_1}=\frac{75^2}{8\times 4{\cdot}24}$$

$$=\mathbf{166}\text{ cm.}$$

5. A carriage spring, centrally loaded and simply supported at the ends, has 10 steel plates, each 5 cm wide by 6 mm thick. If the longest plate is 80 cm long, find the initial radius of curvature of the plates when the greatest bending stress is 1,600 kg/cm² and plates are finally straight.

Neglecting the loss of energy at impact, determine the greatest height from which a mass weighing 20 kg may be dropped centrally on the spring

without exceeding the limiting bending stress of 1,600 kg/cm². **$E=2\times10^6$ kg/cm².** (Lond. Univ.)

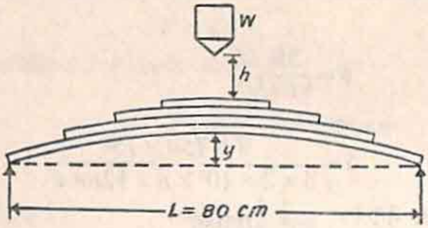

Fig. 13

$$\frac{2f}{t}=\frac{E}{R}$$

$$\therefore \quad R=\frac{Et}{2f}=\frac{2\times10^6\times0{\cdot}6}{2\times1{,}600}$$

$$=375 \text{ cm}$$

$$y=\frac{L^2}{8R}=\frac{6{,}400}{8\times375}=2{\cdot}13 \text{ cm}$$

Potential energy lost by $W=W(h+y)$.

Energy stored in the spring

$$=\frac{f^2}{6E}.V=\frac{f^2}{6E}.\frac{nbtL}{2}$$

$$\therefore \quad W(h+y)=\frac{f^2}{6E}.\frac{nbtL}{2}$$

i.e., $$20(h+2{\cdot}13)=\frac{1{,}600^2\times10\times5\times0{\cdot}6\times80}{6\times2\times10^6\times2}$$

$$\therefore \quad h=10{\cdot}67 \text{ cm.}$$

6. A leaf spring of the semi-elliptic type has 10 plates, each 7·5 cm wide and 1 cm thick. The length of the spring is 120 cm and the plates are of steel having a proof stress (bending) of 6,000 kg/cm². To what radius should the plates be initially bent ?

From what height can a load of 50 kg fall on the centre of the spring if the maximum stress produced is to be one-half of the proof stress ? $E=2\times10^6$ kg/cm².

$$\frac{2f}{t}=\frac{E}{R}$$

$$R=\frac{Et}{2f}=\frac{2\times 10^6\times 1}{2\times 6{,}000}$$

$$=166{\cdot}7\text{ cm}$$

Let P be the equivalent gradually applied load which would produce the same maximum stress as the impact load.

$$f=\frac{3PL}{2nbt^2}$$

i.e., $$3{,}000=\frac{3\times P\times 120}{2\times 10\times 7{\cdot}5}$$

$$\therefore\quad P=1{,}250\text{ kg}$$

The corresponding deflection is

$$\delta=\frac{3PL^3}{8Enbt^3}=\frac{3\times 1{,}250\times 120^3}{8\times 2\times 10^6\times 10\times 7{\cdot}5}$$

$$=5{\cdot}4\text{ cm}$$

Loss of potential energy of the weight

=Gain in strain energy of the spring.

i.e., $$W(h+\delta)=\tfrac{1}{2}P\delta$$

or $$50(h+5{\cdot}4)=\tfrac{1}{2}\times 1{,}250\times 5{\cdot}4$$

$$\therefore\quad h=62{\cdot}1\text{ cm.}$$

7. A load of 20 kg is dropped from a height of 10 cm on to the centre of a carriage spring which is simply supported at its ends. The spring has 10 steel plates each 5 cm wide and 6 mm thick, the longest plate being 75 cm. Calculate the maximum instantaneous stress in the plates and the initial radius of curvature of the spring if the impact just flattens the spring. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

Let P be the equivalent gradually applied load which would cause the same stress as is caused by the impact load.

Then $20(10+y)=\frac{1}{2}\,Py$

$$y=\frac{3PL^3}{8Enbt^3}=\frac{3P\times 75^3}{8\times 2\times 10^6\times 10\times 5\times (0{\cdot}6)^3}$$

$$\therefore\quad P=136{\cdot}5y$$

Hence $$20(10+y)=\tfrac{1}{2}\times 136{\cdot}5y^2$$

or $$y^2-0{\cdot}293y-2{\cdot}93=0$$

Solving $$y=1{\cdot}865\text{ cm}$$

$$P=136{\cdot}5\times 1{\cdot}865=255\text{ kg}$$

$$f=\frac{3PL}{2nbt^2}=\frac{3\times 255\times 75}{2\times 10\times 5\times (0\cdot 6)^2}$$

$$=1{,}594 \text{ kg/cm}^2$$

$$R=\frac{L^2}{8y}=\frac{75^2}{8\times 1\cdot 865}$$

$$=377 \text{ cm.}$$

8. A laminated spring of the quarter-elliptic type is made up of a material 8 cm wide and 6 mm thick. Find the number of leaves required and the maximum stress under the following conditions. The spring is 80 cm long and is to provide for a static deflection of 9 cm under an end load of 200 kg. $E=2\cdot 1\times 10^6$ kg/cm². (Engineering Services, 1961)

$$\delta=\frac{6WL^3}{Enbt^3}$$

i.e.,
$$9=\frac{6\times 200\times 80^3}{2\cdot 1\times 10^6\times n\times 8\times (0\cdot 6)^3}$$

$$\therefore \quad n=18\cdot 8$$

Adopt **19** plates.

$$f=\frac{6WL}{nbt^2}=\frac{6\times 200\times 80}{19\times 8\times (0\cdot 6)^2}$$

$$=1{,}754 \text{ kg/cm}^2.$$

9. A quarter-elliptic leaf spring has 5 plates each 7 cm wide by 1 cm thick with the length of the longest plate 40 cm. If the maximum allowable bending stress is 4,000 kg/cm², find (*a*) the maximum value of the end load and (*b*) the deflection at the end.

Assuming that the spring just straightens under these conditions, determine the initial radius of curvature of the plates. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

$$f=\frac{6WL}{nbt^2}$$

i.e.,
$$4{,}000=\frac{6\times W\times 40}{5\times 7}$$

$$\therefore \quad W=583 \text{ kg}$$

$$\delta=\frac{6WL^3}{Enbt^3}=\frac{6\times 583\times 40^3}{2\times 10^6\times 5\times 7}$$

$$=3\cdot 2 \text{ cm}$$

$$R=\frac{L^2}{2\delta}=\frac{1{,}600}{2\times 3{\cdot}2}$$

$$=\mathbf{250\ cm}$$

Alternatively $$\frac{2f}{t}=\frac{E}{R}$$

$$\therefore \quad R=\frac{Et}{2f}=\frac{2\times 10^6}{2\times 4{,}000}$$

$$=250\text{ cm.}$$

10. A quarter-elliptic, i.e., cantilever, leaf spring has a length of 50 cm and consists of plates each 5 cm wide and 6 mm thick. Find the least number of plates which can be used if the deflection under a gradually applied load of 200 kg is not to exceed 7 cm.

If, instead of being gradually applied, the load of 200 kg falls a distance of 6 mm on to the undeflected spring, find the maximum deflection and stress produced. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

$$\delta=\frac{6WL^3}{Enbt^3}$$

i.e., $$7=\frac{6\times 200\times 50^3}{2\times 10^6\times n\times 5\times (0{\cdot}6)^3}$$

$$\therefore \quad n=9{\cdot}92$$

Adopt **10** plates.

Let P be the equivalent gradually applied load which would produce the same deflection δ_1 as is caused by the impact load.

$$\delta_1=\frac{6PL^3}{Enbt^3}=\frac{6P\times 50^3}{2\times 10^6\times 10\times 5\times (0{\cdot}6)^3}$$

$$\therefore \quad P=28{\cdot}8\delta_1$$

Equating the loss of potential energy of the falling weight to the strain energy stored in the spring

$$200(0{\cdot}6+\delta_1)=\tfrac{1}{2}P\delta_1$$

or $$200(0{\cdot}6+\delta_1)=\tfrac{1}{2}\times 28{\cdot}8\delta_1{}^2$$

or $$\delta_1{}^2-13{\cdot}89\delta_1-8{\cdot}33=0$$

Solving $$\boldsymbol{\delta_1=14{\cdot}47\ \text{cm}}$$

$$P=28{\cdot}8\times 14{\cdot}47=417\text{ kg}$$

Maximum stress, $$f_1=\frac{6PL}{nbt^2}=\frac{6\times 417\times 50}{10\times 5\times (0{\cdot}6)^2}$$

$$=\mathbf{6{,}950\ kg/cm^2.}$$

11. A laminated spring of the quarter-elliptic type, 60 cm long, is to provide a static deflection of 8 cm under an end load of 250 kg. If the leaf material is 6·5 cm wide and 6 mm thick, find the number of leaves required and the maximum stress.

From what height can the load be dropped on to the undeflected spring to cause a maximum stress of 8,000 kg/cm² ? $E=2\times10^6$ kg/cm².

(Lond. Univ.)

$$\delta=\frac{6WL^3}{Enbt^3}$$

i.e.,
$$8=\frac{6\times250\times60^3}{2\times10^6\times n\times6{\cdot}5\times(0{\cdot}6)^3}$$

$$\therefore\quad n=14{\cdot}42$$

Adopt **15** leaves.

$$f=\frac{6WL}{nbt^2}=\frac{6\times250\times60}{15\times6{\cdot}5\times(0{\cdot}6)^2}$$

$$=\mathbf{2{,}560\ kg/cm^2}$$

Let P be the equivalent gradually applied load to cause a maximum stress of 8,000 kg/cm².

$$8{,}000=\frac{6P\times60}{15\times6{\cdot}5\times(0{\cdot}6)^2}$$

$$\therefore\quad P=780\text{ kg}$$

The corresponding deflection

$$\delta_1=\frac{6\times780\times60^3}{2\times10^6\times15\times6{\cdot}5\times(0{\cdot}6)^3}$$

$$=24\text{ cm}$$

Loss of potential energy = Gain of strain energy.

i. e.,
$$250(h+24)=\tfrac{1}{2}\times780\times24$$

$$\therefore\quad h=\mathbf{13{\cdot}44\ cm.}$$

12. A flat spiral spring is made of steel 12 mm broad and 0·5 mm thick, the length of spiral being 6 metres. Determine (*a*) the maximum turning moment which can be applied to the spindle if the stress is not to exceed 6,000 kg/cm², (*b*) the number of turns then given to the spindle, and (*c*) the energy stored. $E=2{\cdot}1\times10^6$ kg/cm². (Lond. Univ.)

$$f=\frac{12M}{bt^2}$$

i. e.,
$$6{,}000=\frac{12M}{1{\cdot}2\times(0{\cdot}05)^2}$$

$$\therefore \quad M = 1{\cdot}5 \text{ kg cm}$$

$$\theta = \frac{ML}{EI} = \frac{1{\cdot}5 \times 600 \times 12}{2{\cdot}1 \times 10^6 \times 1{\cdot}2 \times (0{\cdot}05)^3}$$

$$= 34{\cdot}3 \text{ radians}$$

Number of turns given to the spindle

$$= \frac{34{\cdot}3}{2\pi} = 5{\cdot}46$$

Energy stored, $\quad U = \frac{1}{2} M\theta = \frac{1{\cdot}5 \times 34{\cdot}3}{2}$

$$= 25{\cdot}7 \text{ kg cm.}$$

13. Find the length of a flat spiral spring 25 mm wide by 0·5 mm thick to store 80 kg cm of energy for a limiting stress of 8,000 kg/cm². Find also the torque required, and the number of turns of the winding spindle to wind up the spring. $E = 2{\cdot}1 \times 10^5$ kg/cm². (Lond. Univ.)

$$f = \frac{12M}{bt^2}$$

i. e., $\quad 8{,}000 = \frac{12M}{2{\cdot}5 \times (0{\cdot}05)^2}$

$$\therefore \quad M = \tfrac{25}{6} = 4{\cdot}17 \text{ kg cm}$$

$$U = \tfrac{1}{2} M\theta$$

i. e., $\quad 80 = \frac{1}{2} \times \frac{25}{6}\, \theta$

$$\therefore \quad \theta = 38{\cdot}4 \text{ radians}$$

Number of turns given to the spindle

$$= \frac{38{\cdot}4}{2\pi} = 6{\cdot}11$$

$$\theta = \frac{ML}{EI}$$

i. e., $\quad 38{\cdot}4 = \frac{25 \times L \times 12}{6 \times 2{\cdot}1 \times 10^6 \times 2{\cdot}5 \times (0{\cdot}05)^3}$

$$\therefore \quad L = 504 \text{ cm} = 5{\cdot}04 \text{ m.}$$

14. A flat spiral spring is made of steel 12 mm wide, 0·6 mm thick and 3 metres long. Assuming that the spring is in an unstressed condition, determine the maximum stress produced and the amount of energy stored in the spring by three complete turns of the spindle.

$E = 2{\cdot}1 \times 10^6$ kg/cm².

$$\theta = \frac{ML}{EI}$$

i. e., $$6\pi = \frac{M \times 300 \times 12}{2{\cdot}1 \times 10^6 \times 1{\cdot}2 \times (0{\cdot}06)^3}$$

$$\therefore \quad M = 2{\cdot}85 \text{ kg cm}$$

$$f = \frac{12M}{bt^2} = \frac{12 \times 2{\cdot}85}{1{\cdot}2 \times (0{\cdot}06)^2}$$

$$= 7{,}920 \text{ kg/cm}^2$$

$$U = \tfrac{1}{2}M\theta = \tfrac{1}{2} \times 2{\cdot}85 \times 6\pi$$

$$= 26{\cdot}9 \text{ kg cm.}$$

15. A conical spiral spring is submitted to the action of an axial force P. Determine the safe magnitude of P if the allowable stress in shear is 1,200 kg/cm², diameter of the cross-section $d=3$ cm, radius of the cone at the top of the spring $R_1=5$ cm and at the bottom $R_2=20$ cm. Determine also the extension of the spring if the number of coils $n=8$ and $C=8\times10^5$ kg/cm².

Maximum torque, $$T = PR_2 = 20P$$

$$T = \frac{\pi d^3}{16} \times q$$

or $$20P = \frac{\pi \times 27}{16} \times 1{,}200$$

$$\therefore \quad P = 318 \text{ kg}$$

$$\delta = \frac{16Pn}{Cd^4}\,(R_1^2 + R_2^2)(R_1 + R_2)$$

$$= \frac{16 \times 318 \times 8 \times 425 \times 25}{8 \times 10^5 \times 81}$$

$$= 6{\cdot}67 \text{ cm.}$$

CHAPTER IV

THICK CYLINDERS AND SPHERES

1. A thick cylinder, 50 cm external diameter and 40 cm internal diameter, is subjected simultaneously to internal and external pressures. If the internal pressure is 250 kg/cm^2 and the hoop stress at the inside of the cylinder is 450 kg/cm^2 (tensile), determine the intensity of the external pressure. (Engineering Services, 1954)

$$p \text{ compression} = \frac{B}{r^2} - A$$

$$f \text{ tension} = \frac{B}{r^2} + A$$

At $r = 20$ cm, $p = 250$ kg/cm^2

$$\therefore \quad 250 = \frac{B}{400} - A \qquad (1)$$

At $r = 20$ cm, $f = 450$ kg/cm^2

$$\therefore \quad 450 = \frac{B}{400} + A \qquad (2)$$

From equations 1 and 2

$$A = 100, \ B = 140{,}000$$

At $r = 25$ cm, $p = \dfrac{B}{25^2} - A$

$$= \frac{140{,}000}{625} - 100$$

$$= \mathbf{124\ kg/cm^2}.$$

2. The internal and external diameters of a thick hollow cylinder are 8 cm and 12 cm respectively. It is subjected to an external pressure of 400 kg/cm^2 when the internal pressure is 1,200 kg/cm^2. Calculate the circumferential stress at the external and internal surfaces and determine the radial and circumferential stresses at the mean radius. Plot the radial and circumferential stresses on the thickness of the cylinder as base. (Engineering Services, 1968)

$$p = \frac{B}{r^2} - A$$

At $r=6$ cm, $400=\frac{B}{36}-A$ .. (1)

At $r=4$ cm, $1{,}200=\frac{B}{16}-A$ .. (2)

Solving $A=240,\ B=23{,}040$

$$f=\frac{B}{r^2}+A$$

At $r=6$ cm, $f=\frac{23{,}040}{36}+240$

$=\mathbf{880\ kg/cm^2}$

At $r=4$ cm, $f=\frac{23{,}040}{16}+240$

$=\mathbf{1{,}680\ kg/cm^2}$

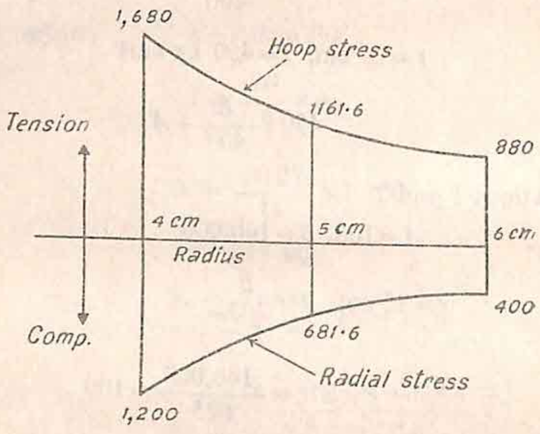

Fig. 14

At $r=5$ cm, $p=\frac{23{,}040}{25}-240$

$=\mathbf{681{\cdot}6\ kg/cm^2}$

At $r=5$ cm, $f=\frac{23{,}040}{25}+240$

$=\mathbf{1161{\cdot}6\ kg/cm^2}$.

The stress diagram is shown in Fig. 14.

3. The cylinder of a hydraulic press has an internal diameter of 30 cm and is to be designed to withstand a pressure of 100 kg/cm² without the material being stressed over 200 kg/cm². Determine the thickness of the metal and the stress on the outside of the cylinder.

Sketch a diagram showing the variation of radial and hoop stresses across the thickness of the wall of the cylinder. (Engineering Services, 1959)

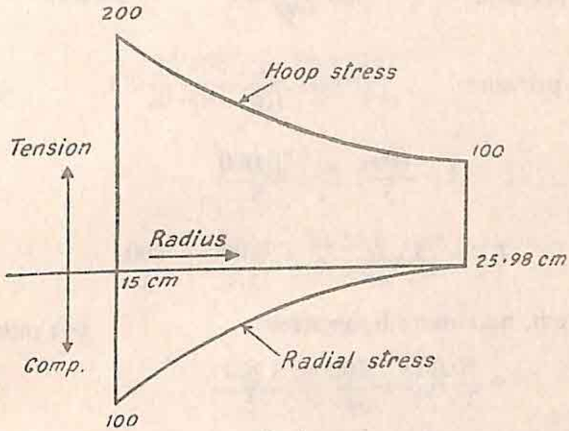

Fig. 15

Let R_1 be the external radius.

Internal pressure $\qquad 100=\dfrac{B}{225}-A \qquad \ldots \qquad (1)$

Maximum hoop stress at the inside

$$200=\frac{B}{225}+A \qquad \ldots \qquad (2)$$

Solving $\qquad A=50,\ B=33{,}750$

External pressure $\qquad 0=\dfrac{B}{R_1^2}-A$

i. e., $\qquad 0=\dfrac{33{,}750}{R_1^2}-50$

or $\qquad R_1^2=675$

$$\therefore \quad R_1=25{\cdot}98 \text{ cm}$$

Thickness of metal $=25{\cdot}98-15=$ **10·98 cm**

At the outside of the cylinder

$$f=\frac{B}{R_1^2}+A=\frac{33{,}750}{675}+50$$

$$=\mathbf{100\ kg/cm^2}$$

The stress diagram is shown in Fig. 15.

4. A thick cylinder of steel having an internal diameter of 10 cm and an external diameter of 20 cm is subjected to an internal pressure of 800 kg/cm² and an external pressure of 100 kg/cm². Find the maximum direct and shearing stresses in the cylinder and calculate the change of external diameter. $E=2\times10^6$ kg/cm² and Poisson's ratio$=0{\cdot}3$. (Lond. Univ.)

Internal pressure $$800=\frac{B}{5^2}-A \qquad \text{(1)}$$

External pressure $$100=\frac{B}{10^2}-A \qquad \text{(2)}$$

Solving $$A=\frac{400}{3},\ B=\frac{70{,}000}{3}$$

$$f=\frac{B}{r^2}+A=\frac{70{,}000}{3r^2}+\frac{400}{3}$$

At $r=5$ cm, maximum hoop stress

$$=\frac{70{,}000}{3\times25}+\frac{400}{3}=\frac{3{,}200}{3}$$

$$=\mathbf{1{,}067\ kg/cm^2}$$

Maximum shear stress

$$=\frac{1}{2}\left(\frac{3{,}200}{3}+800\right)=\frac{2{,}800}{3}$$

$$=\mathbf{933\ kg/cm^2}$$

At $$r=10\text{ cm},\ f=\frac{70{,}000}{3\times100}+\frac{400}{3}$$

$$=\frac{1{,}100}{3}$$

Hoop strain at the outer surface

$$=\frac{1}{E}\left(\frac{1{,}100}{3}+0{\cdot}3\times100\right)=\frac{1{,}190}{3E}$$

Increase of external diameter

$$=\frac{1{,}190}{3\times2\times10^6}\times20$$

$$=0{\cdot}00397\text{ cm.}$$

5. The cylinder of a hydraulic ram is 16 cm internal diameter. Find the thickness required to withstand an internal pressure of 600 kg/cm², if the maximum tensile stress is limited to 900 kg/cm² and the maximum shear stress to 800 kg/cm².

Let R be the external radius.

Internal pressure $$600=\frac{B}{8^2}-A \quad \ldots \quad (1)$$

External pressure $$0=\frac{B}{R^2}-A \quad \ldots \quad (2)$$

Solving $$A=\frac{38{,}400}{R^2-64},\ B=\frac{38{,}400R^2}{R^2-64}$$

$$f=\frac{B}{r^2}+A$$

$$=\frac{38{,}400}{(R^2-64)}\left(\frac{R^2}{r^2}+1\right)$$

Maximum f at $$r=8 \text{ cm},$$

$$f_2=\frac{R^2+64}{R^2-64}\times 600$$

i. e., $$900=\frac{R^2+64}{R^2-64}\times 600$$

or $$3R^2-192=2R^2+128$$

or $$R^2=320$$

$$\therefore \quad R=17{\cdot}89 \text{ cm}$$

Maximum shear stress

$$=\tfrac{1}{2}(f_2+600)=\frac{600}{2}\left(\frac{R^2+64}{R^2-64}+1\right)$$

$$=\frac{600R^2}{R^2-64}$$

i.e., $$800=\frac{600R^2}{R^2-64}$$

or $$4R^2-256=3R^2$$

or $$R^2=256$$

$$\therefore \quad R=16 \text{ cm}$$

Thickness $=17{\cdot}89-8=9{\cdot}89$ cm.

(7) If a cylinder of internal diameter d, wall thickness t and subjected to internal pressure only, is assumed to be a thin shell, what is the great-

est value for the ratio $\frac{t}{d}$ if the error in the estimated maximum hoop stress is not to exceed 5 percent ?

(*b*) A cylinder having inside and outside diameters of 16 cm and 20 cm respectively has been designed to withstand a certain internal pressure but reboring becomes necessary. Determine the limit to the new inside diameter if the maximum hoop stress is not to exceed the previous value by more than 5 percent while the internal pressure is the same as before. Treat the cylinder as "thick-walled." (Lond. Univ.)

(a) Maximum hoop stress

$$f_2 = \frac{R^2+r^2}{R^2-r^2}\cdot p = \frac{D^2+d^2}{D^2-d^2}\cdot p,$$

where D and d are the external and internal diameters respectively.

or
$$f_2 = \frac{(d+2t)^2+d^2}{(d+2t)^2-d^2}\cdot p$$

$$= \frac{d^2+2dt \div 2t^2}{2dt+2t^2}\cdot p$$

Let
$$\frac{t}{d} = n$$

Then
$$f_2 = \frac{d^2+2d^2n+2d^2n^2}{2d^2n+2d^2n^2}\cdot p$$

$$= \frac{1+2n+2n^2}{2n+2n^2}\cdot p$$

From thin cylinder formula

$$f = \frac{pd}{2t} = \frac{p}{2n}$$

Given
$$\frac{f_2-f}{f_2} = 0{\cdot}05$$

or
$$1-\frac{f}{f_2} = 0{\cdot}05$$

or
$$\frac{f}{f_2} = 0{\cdot}95$$

or
$$\frac{2n+2n^2}{2n(1+2n+2n^2)} = 0{\cdot}95$$

or $$\frac{1+n}{1+2n+2n^2}=0{\cdot}95$$

or $$1{\cdot}9n^2+0{\cdot}9n-0{\cdot}05=0$$

Solving $$n=0{\cdot}0502.$$

(b) Initially

$$f_2=\frac{R^2+r^2}{R^2-r^2}.p$$

$$=\frac{100+64}{100-64}\,p=\frac{41}{9}p$$

Let r_1 be the internal radius after reboring.

$$f_2'=\frac{R^2+r_1^2}{R^2-r_1^2}.p=\frac{100+r_1^2}{100-r_1^2}.p$$

Given $$f_2'=1{\cdot}05\,f_2$$

i. e., $$\frac{100+r_1^2}{100-r_1^2}.p=1{\cdot}05\times\frac{41}{9}p$$

or $$9(100+r_1^2)=43{\cdot}05(100-r_1^2)$$

or $$52{\cdot}05r_1^2=3{,}405$$

or $$r_1^2=65{\cdot}4$$

$$\therefore\quad r_1=8{\cdot}09\text{ cm}$$

Inside diameter $=2r_1=16{\cdot}18$ cm.

7. A steel cylinder 20 cm and 15 cm external and internal diameter respectively is used for a working pressure of 120 kg/cm². Owing to external corrosion the external diameter of the cylinder has to be machined down to 19·5 cm. Find by how much the working pressure must be reduced for the maximum hoop stress to be the same as before.

(Ans. 10·1 kg/cm²)

8. Two thick steel cylinders *A* and *B*, closed at the ends, have the same dimensions, the outside diameter being 1·6 times the inside. *A* is subjected to internal pressure only and *B* to external pressure only. Find the ratio of these pressures (*a*) when the greatest circumferential stress has the same numerical value, and (*b*) when the greatest circumferential strain has the same numerical value. Poisson's ratio = 0·304. (Lond. Univ.)

Cylinder A :

Internal pressure $$p_2=\frac{B}{R_2^2}-A \qquad \text{.. (1)}$$

External pressure $$0=\frac{B}{R_1^2}-A \qquad \text{.. (2)}$$

Solving, $$A=\frac{R_2^2}{R_1^2-R_2^2}.p_2,\ B=\frac{R_1^2R_2^2}{R_1^2-R_2^2}.p_2$$

$$f=\frac{B}{r^2}+A=\frac{R_2^2}{R_1^2-R_2^2}.p_2\left(\frac{R_1^2}{r^2}+1\right)$$

Greatest circumferential stress at $r=R_2$,

$$f_2=\frac{R_2^2}{R_1^2-R_2^2}.p_2\left(\frac{R_1^2}{R_2^2}+1\right)$$

$$=\frac{R_1^2+R_2^2}{R_1^2-R_2^2}.p_2$$

Given $$R_1=1{\cdot}6R_2$$

$$\therefore\quad f_2=\frac{(1{\cdot}6R_2)^2+R_2^2}{(1{\cdot}6R_2)^2-R_2^2}.p_2=\frac{3{\cdot}56}{1{\cdot}56}p_2 \text{ tensile}$$

Longitudinal stress f_l is given by the equilibrium equation.

$$f_l.\pi(R_1^2-R_2^2)=p_2.\pi R_2^2$$

or $$f_l=\frac{R_2^2}{R_1^2-R_2^2}.p_2=\frac{R_2^2}{(1{\cdot}6R_2)^2-R_2^2}.p_2$$

$$=\frac{p_2}{1{\cdot}56} \text{ tensile}$$

Greatest circumferential strain at $r=R_2$,

$$=\frac{1}{E}\left(f_2+\frac{p_2}{m}-\frac{f_l}{m}\right)$$

$$=\frac{p_2}{E}\left(\frac{3{\cdot}56}{1{\cdot}56}+0{\cdot}304-\frac{0{\cdot}304}{1{\cdot}56}\right)$$

$$=\frac{2{\cdot}391p_2}{E}$$

Cylinder B :

External pressure $$p_1=\frac{B'}{R_1^2}-A' \qquad \ldots (3)$$

Internal pressure $$0=\frac{B'}{R_2^2}-A' \qquad \ldots (4)$$

Solving $$A'=-\frac{R_1^2}{R_1^2-R_2^2}p_1,\ B'=-\frac{R_1^2R_2^2}{R_1^2-R_2^2}p$$

$$f=\frac{B'}{r^2}+A=-\frac{R_1^2}{R_1^2-R_2^2}p_1\left(\frac{R_2^2}{r^2}+1\right)$$

Greatest circumferential stress at $r=R_2$,

$$f_2'=-\frac{2R_1^2}{R_1^2-R_2^2}.p_1=-\frac{2\times(1{\cdot}6R_2)^2}{(1{\cdot}6R_2)^2-R_2^2}.p_1$$

$$=-\frac{5{\cdot}12}{1{\cdot}56}p_1 \text{ compressive}$$

Longitudinal stress f_l' is given by the equilibrium equation.

$$f_l'.\pi(R_1^2-R_2^2)=p_1.\pi R_1^2$$

$$\therefore\quad f_l'=\frac{R_1^2}{R_1^2-R_2^2}.p_1=\frac{(1{\cdot}6R_2)^2}{(1{\cdot}6R_2)^2-R_2^2}.p_1$$

$$=\frac{2{\cdot}56}{1{\cdot}56}p_1 \text{ compressive}$$

Greatest circumferential strain at $r=R_2$,

$$=\frac{1}{E}\left(-\frac{5{\cdot}12}{1{\cdot}56}p_1+0{\cdot}304\times\frac{2{\cdot}56}{1{\cdot}56}p_1\right)$$

$$=-\frac{2{\cdot}783p_1}{E}$$

Case (a):

$$\frac{3{\cdot}56}{1{\cdot}56}p_2=\frac{5{\cdot}12}{1{\cdot}56}p_1$$

or
$$\frac{p_2}{p_1}=\frac{5{\cdot}12}{3{\cdot}56}=\mathbf{1{\cdot}44}$$

Case (b):

$$\frac{2{\cdot}391p_2}{E}=\frac{2{\cdot}783p_1}{E}$$

or
$$\frac{p_2}{p_1}=\frac{2{\cdot}783}{2{\cdot}391}=\mathbf{1{\cdot}16}.$$

9. A steel shaft, originally 10 cm diameter, is subjected to a uniform radial compressive stress of 200 kg/cm². Assuming the radial stress remains constant, find the uniform longitudinal stress required to reduce the initial diameter by 0·0012 cm, and calculate the alteration of volume for a 15 cm length of shaft. $E=2\times10^6$ kg/cm²; Poisson's ratio$=0{\cdot}304$.

(Lond. Univ.)

$$f=\frac{B}{r^2}+A$$

$$p=\frac{B}{r^2}-A$$

But since the stresses cannot be infinite at the centre of the shaft (i. e., $r=0$), then B must be zero, i. e.,

$$f=A=-p$$

which means that the hoop stress is compressive and equal to the radial stress (and consequently the external pressure), both stresses being constant throughout.

$f=-p=-200$ kg/cm², both compressive.

Let the longitudinal stress be f_l tension.

Hoop strain $\quad e_h=\frac{1}{E}\left(-200+\frac{200}{m}-\frac{f_l}{m}\right)$

$$=\frac{1}{E}(-200+0{\cdot}304\times200-0{\cdot}304f_l)$$

$$=-\frac{1}{E}(139{\cdot}2+0{\cdot}304f_l)$$

i. e., $\quad \frac{1}{2\times10^6}(139{\cdot}2+0{\cdot}304f_l)=\frac{0{\cdot}0012}{10}$

$\therefore\ f_l=$ **332 kg/cm² tension**

Longitudinal strain e_l

$$=\frac{1}{E}\left(f_l+\frac{200}{m}+\frac{200}{m}\right)$$

$$=\frac{1}{2\times10^6}(332+0{\cdot}304\times400)$$

$$=0{\cdot}0002268$$

Hoop strain $\quad e_h=-\frac{0{\cdot}0012}{10}=-0{\cdot}00012$

Volumetric strain $=2e_h+e_l=-0{\cdot}0000132$

Decrease in volume $=0{\cdot}0000132\times\frac{\pi}{4}\times10^2\times15$

$=$ **$0{\cdot}01555$ cm³.**

10. Find the ratio of thickness to internal diameter for a tube subjected to internal pressure when the ratio of pressure to maximum circumferential stress is 0·5.

Find the alteration of thickness of metal in such a tube 20 cm internal diameter when the pressure is 750 kg/cm². $E=2\times10^6$ kg/cm², Poisson's ratio = 0·3. (Lond. Univ.)

Maximum circumferential stress at the inside,

$$f_2=\frac{R_1^2+R_2^2}{R_1^2-R_2^2}\cdot p_2$$

But $$\frac{p_2}{f_2}=0{\cdot}5$$

$$\therefore \quad R_1^2-R_2^2=0{\cdot}5(R_1^2+R_2^2)$$

or $$R_1^2=3R_2^2$$

$$\therefore \quad R_1=\sqrt{3}R_2$$

Thickness $$t=R_1-R_2=(\sqrt{3}-1)R_2$$

$$\therefore \quad \frac{t}{D_2}=\frac{(\sqrt{3}-1)R_2}{2R_2}=0{\cdot}366$$

$$R_2=10 \text{ cm}, \ R_1=\sqrt{3}R_2=17{\cdot}32 \text{ cm}$$

Longitudinal stress f_l is given by the equilibrium equation.

$$f_l.\pi(R_1^2-R_2^2)=p_2.\pi R_2^2$$

$$\therefore \quad f_l=\frac{R_2^2}{R_1^2-R_2^2}\cdot p_2=\frac{R_2^2}{3R_2^2-R_2^2}\times 750$$

$$=375 \text{ kg/cm}^2 \text{ tension}$$

$$f_2=2p_2=1{,}500 \text{ kg/cm}^2 \text{ tension}$$

Hoop strain at the inner surface

$$=\frac{1}{E}(1{,}500+0{\cdot}3\times750-0{\cdot}3\times375)$$

$$=\frac{1612{\cdot}5}{E}$$

Increase in internal radius

$$=\frac{1612{\cdot}5}{E}\times10=\frac{16{,}125}{E} \text{ cm}$$

$$f-p=\text{constant}=2A$$

$$f_2-p_2=1{,}500-750=750 \text{ kg/cm}^2$$

$$p_1=0$$

$$\therefore \quad f_1=750 \text{ kg/cm}^2 \text{ tension}$$

Hoop strain at the outer surface

$$=\frac{1}{E}(750-0\cdot3\times375)=\frac{637\cdot5}{E}$$

Increase in external radius

$$=\frac{637\cdot5}{E}\times17\cdot32=\frac{11{,}040}{E}$$

Decrease in thickness

$$=\frac{16{,}125-11{,}040}{E}=\frac{5{,}085}{2\times10^6}$$

$$=0\cdot00254 \text{ cm.}$$

11. A steel tube is 3 cm internal diameter and 5 mm thick. One endis closed and the other end is screwed into a pressure vessel. The projecting length is 30 cm. Neglecting any constraints due to the ends, calculate the safe internal pressure if the allowable stress is not to exceed 1,400 kg/cm². Calculate the increase in the internal volume under this pressure. $E=2\times10^6$ kg/cm². Poisson's ratio $=\frac{1}{3\cdot5}$. (Lond. Univ.)

External pressure, $0=\frac{B}{4}-A$ .. (1)

Maximum hoop stress at the inner surface,

$$1{,}400=\frac{4B}{9}+A \quad .. \quad (2)$$

Solving $A=504,\ B=2{,}016$

Internal pressure, $p=\frac{4B}{9}-A$

$$=392 \text{ kg/cm}^2$$

Longitudinal stress, $f_l=\frac{R_2^2}{R_1^2-R_2^2}\cdot p=\frac{1\cdot5^2\times392}{2^2-1\cdot5^2}$

$$=504 \text{ kg/cm}^2 \text{ tension}$$

Hoop strain, $e_h=\frac{1}{E}\left(1{,}400+\frac{392}{3\cdot5}-\frac{504}{3\cdot5}\right)$

$$=\frac{1{,}368}{E}$$

Longitudinal strain, $e_l = \frac{1}{E}\left(504 - \frac{1,400}{3\cdot5} + \frac{392}{3\cdot5}\right)$

$$= \frac{216}{E}$$

Volumetric strain $= 2e_h + e_l$

$$= 2 \times \frac{1,368}{E} + \frac{216}{E}$$

$$= \frac{2,952}{E}$$

Internal volume $= \frac{\pi}{4} \times 3^2 \times 30 \text{ cm}^3$

$\therefore$ Increase in internal volume

$$= \frac{2,952}{2 \times 10^6} \times \frac{\pi}{4} \times 3^2 \times 30 = 0\cdot313 \text{ cm}^3$$

12. A tube is 1 cm diameter bore and 1·5 cm external diameter. It is 24 cm long and is closed at one end. Fluid is forced into the open end under a pressure of 300 kg/cm² when the tube is full of fluid. Calculate the excess volume of fluid forced into the tube beyond what is required to fill it at zero pressure. E for tube $= 6 \times 10^5$ kg/cm², Poisson's ratio $= 0\cdot25$; modulus of cubic compressibility for the fluid $= 10,000$ kg/cm².

(Lond. Univ.)

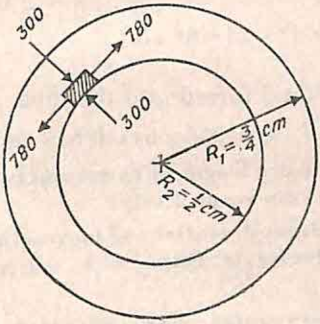

Fig. 16

At $r = \frac{1}{2}$ cm, $300 = 4B - A$ .. (1)

At $r = \frac{3}{4}$ cm, $0 = \frac{16B}{9} - A$ .. (2)

Solving $A = 240, \quad B = 135$

At $r = \frac{1}{2}$ cm, $f_2 = 4B + A = 780$ kg/cm²

Hoop strain, $e_h = \dfrac{1}{E}(780 + 0{\cdot}25 \times 300)$

$$= \frac{855}{E}$$

Longitudinal strain, $e_l = \dfrac{1}{E}(-0{\cdot}25 \times 780 + 0{\cdot}25 \times 300)$

$$= -\frac{120}{E}$$

Volumetric strain of tube,

$$\frac{\delta V}{V} = 2e_h + e_l = 2 \times \frac{855}{E} - \frac{120}{E}$$

$$= \frac{1{,}590}{E} = \frac{1{,}590}{6 \times 10^5}$$

$$= 0{\cdot}00265$$

Volumetric strain of water,

$$\frac{\delta V}{V} = \frac{p}{K} = \frac{300}{10{,}000} = 0{\cdot}03$$

Total $\dfrac{\delta V}{V} = 0{\cdot}00265 + 0{\cdot}03 = 0{\cdot}03265$

Volume $V = \dfrac{\pi}{4} \times 1^2 \times 24 = 6\pi \text{ cm}^3$

Excess volume of fluid forced into the tube

$$= 0{\cdot}03265 \times 6\pi = 0{\cdot}615 \text{ cm}^3.$$

13. A steel cylinder has a length of 25 cm and internal and external diameters of 10 cm and 14 cm respectively.

(*a*) Determine the circumferential and longitudinal stresses at the inner surface when the cylinder is filled with water under a pressure of 100 kg/cm².

(*b*) How much more water does the cylinder contain than that required to fill at atmospheric pressure ?

Take *E* for steel $= 2 \times 10^6$ kg/cm², Poisson's ratio for steel $= 0{\cdot}3$ and *K* for water $= 2 \times 10^4$ kg/cm².

At $r = 5$ cm, $p = 100$

$$\therefore \quad 100 = \frac{B}{25} - A \qquad \ldots \quad (1)$$

At $r=7$ cm, $p=0$

$$\therefore \quad 0=\frac{B}{49}-A \qquad \ldots \qquad (2)$$

Solving $A=\frac{625}{6}$, $B=\frac{30,625}{6}$

At $r=5$ cm, $f=\frac{B}{25}+A$

$$=\frac{30,625}{6\times 25}+\frac{625}{6}=\frac{925}{3}$$

$$=\mathbf{308\ kg/cm^2\ tension}$$

The longitudinal stress f_l is given by the equilibrium equation.

$$\pi(7^2-5^2).f_l=\pi\times 5^2\times 100$$

$$\therefore \quad f_l=\frac{625}{6}=\mathbf{104{\cdot}2\ kg/cm^2\ tension}$$

Hoop strain, $e_h=\frac{1}{E}\left(\frac{925}{3}+0{\cdot}3\times 100-0{\cdot}3\times\frac{625}{6}\right)$

$$=\frac{307}{E}$$

Longitudinal strain, $e_l=\frac{1}{E}\left(\frac{625}{6}+0{\cdot}3\times 100-0{\cdot}3\times\frac{925}{3}\right)$

$$=\frac{41{\cdot}7}{E}$$

Volumetric strain $\frac{\delta V}{V}$ for cylinder

$$=2e_h+e_l=\frac{1}{2\times 10^6}(2\times 307+41{\cdot}7)$$

$$=0{\cdot}000328$$

Increase in volume of cylinder $=0{\cdot}000328V$ $\qquad$ .. (3)

For water, $\frac{\delta V}{V}=\frac{p}{K}=\frac{100}{2\times 10^4}$

$$=0{\cdot}005$$

Decrease in volume of water $=0{\cdot}005V$ $\qquad$ .. (4)

Additional volume of water $=(3)+(4)$

$$=0{\cdot}005328\,V=0{\cdot}005328\times\frac{\pi}{4}\times10^2\times25$$

$$=10{\cdot}46\ \text{cm}^3$$

14. A thick steel tube with closed ends, of inside and outside diameters 5 cm and 7 cm respectively, contains oil at a pressure of 10 kg/cm². The oil is allowed to escape until the pressure in the tube has fallen to 7·5 kg/cm². Find how much oil has been released per metre length of tube, if bending due to end effects is negligible. E for steel $=2\times10^6$ kg/cm²; Poisson's ratio for steel $=0{\cdot}25$; K for oil $=2{\cdot}75\times10^4$ kg/cm². (Lond. Univ.)

(Ans. $0{\cdot}195$ cm³)

15. A steel hoop, of 20 cm outer and 13 cm inner diameters, is shrunk on a hollow steel cylinder of 8 cm inner diameter, the pressure of shrinkage being 200 kg/cm².

When subjected to internal fluid pressure of 700 kg/cm², what will be

(*a*) the greatest circumferential stress induced in the cylinder,

(*b*) the radial pressure between the cylinder and the hoop, and

(*c*) the greatest circumferential stress in the hoop ?

Assume that the stresses induced are within the proportional limit.

(Engineering Services, 1968)

Due to shrinkage :

Hoop :

At $\quad r=\frac{13}{2}$ cm, $\quad p=200=\dfrac{4B}{169}-A$

At $\quad r=10$ cm, $\quad p=0=\dfrac{B}{100}-A$

Solving $\quad A=\dfrac{33{,}800}{231}\quad B=\dfrac{3{,}380{,}000}{231}$

At $\quad r=\frac{13}{2}$ cm, $\quad f=\dfrac{4B}{169}+A$

$$=\frac{4\times3{,}380{,}000}{169\times231}+\frac{33{,}800}{231}$$

$$=493\ \text{kg/cm}^2\ \text{tension} \qquad \ldots \quad (1)$$

Cylinder :

At $\quad r=\frac{13}{2}$ cm, $\quad p=200=\dfrac{4B'}{169}-A'$

At $$r=4 \text{ cm},\ p=0=\frac{B'}{16}-A'$$

Solving $$A'=-\frac{6{,}760}{21}\quad B'=-\frac{108{,}160}{21}$$

At $$r=4 \text{ cm},\ f=\frac{B'}{16}+A'$$

$$=-\frac{108{,}160}{16\times 21}-\frac{6{,}760}{21}$$

$$=-644 \text{ kg/cm}^2 \text{ comp.} \qquad \cdots \quad (2)$$

Due to internal pressure :

At $$r=4 \text{ cm},\ p=700=\frac{B''}{16}-A''$$

At $$r=10 \text{ cm},\ p=0=\frac{B''}{100}-A''$$

Solving $$A''=\frac{400}{3},\ B''=\frac{40{,}000}{3}$$

At $$r=4 \text{ cm},\ f=\frac{B''}{16}+A''$$

$$=\frac{40{,}000}{16\times 3}+\frac{400}{3}$$

$$=967 \text{ kg/cm}^2 \text{ tension} \qquad \cdots \quad (3)$$

At $$r=\tfrac{13}{2} \text{ cm},\ f=\frac{4B''}{169}+A''$$

$$=\frac{4\times 40{,}000}{169\times 3}+\frac{400}{3}$$

$$=449 \text{ kg/cm}^2 \text{ tension} \qquad \cdots \quad (4)$$

At $$r=\tfrac{13}{2} \text{ cm},\ p=\frac{4B''}{169}-A''$$

$$=\frac{4\times 40{,}000}{169\times 3}-\frac{400}{3}$$

$$=182 \text{ kg/cm}^2$$

Greatest circumferential stress in the cylinder

$$=(2)+(3)=-644+967$$

$$=\mathbf{323\ kg/cm^2\ tension}$$

Greatest circumferential stress in the hoop

$$=(1)+(4)=493+449$$

$$=\mathbf{942\ kg/cm^2\ tension}$$

Radial pressure between cylinder and hoop

$$=200+182=\mathbf{382\ kg/cm^2}.$$

16. A compound tube is made by shrinking one tube on another, the final dimensions being; internal diameter, 8 cm; external diameter, 16 cm; diameter at the junction of the tubes, 12 cm. If the radial pressure at the common 6-cm radius is 150 kg/cm² find the greatest hoop tension and hoop pressure in the compound cylinder. What difference must there be in the external diameter of the inner tube and the internal diameter of the outer tube before shrinking on, and what is the least difference of temperature necessary to allow the outer one passing over the inner one ? Take the coefficient of expansion as $11{\cdot}2\times10^{-6}$ per degree C, and $E=2\times10^6$ kg/cm².

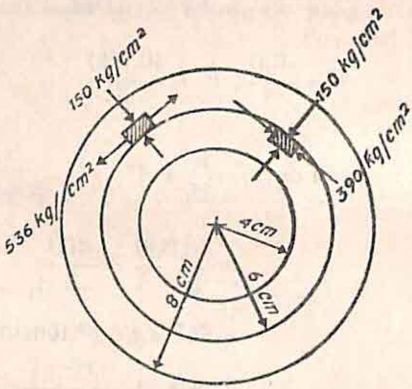

Fig. 17

Outer tube :

At $\qquad r=6\text{ cm},\ p=150=\dfrac{B}{36}-A$

At $\qquad r=8\text{cm},\quad p=0=\dfrac{B}{64}-A$

Solving $\qquad A=\dfrac{1{,}350}{7},\ B=\dfrac{86{,}400}{7}$

At $r=6$ cm, maximum hoop tension

$$=\frac{B}{36}+A=\frac{86{,}400}{36\times 7}+\frac{1{,}350}{7}$$

$$=\mathbf{536\ kg/cm^2}$$

Hoop strain at the inner surface

$$=\frac{1}{E}\left(536+\frac{150}{m}\right)$$

Increase in the internal diameter

$$=\frac{12}{E}\left(536+\frac{150}{m}\right) \quad .. \quad (1)$$

Inner tube :

At $r=6$ cm, $p=150=\frac{B'}{36}-A'$

At $r=4$ cm, $p=0=\frac{B'}{16}-A'$

Solving $A'=-270,\ B'=-4{,}320$

Maximum hoop pressure at $r=4$ cm

$$=\frac{B'}{16}+A'=-\frac{4{,}320}{16}-270$$

$$=\mathbf{-540\ kg/cm^2\ comp.}$$

At $r=6$ cm, $f=\frac{B'}{36}+A'$

$$=-\frac{4{,}320}{36}-270$$

$$=-390\ \text{kg/cm}^2\ \text{comp.}$$

Hoop strain at the outer surface

$$=-\frac{1}{E}\left(390-\frac{150}{m}\right)$$

Decrease in the external diameter

$$=\frac{12}{E}\left(390-\frac{150}{m}\right) \quad .. \quad (2)$$

Required difference in diameters

$$=(1)+(2)=\frac{12}{E}(536+390)$$

$$=\frac{12\times926}{2\times10^{6}}$$

$$=0{\cdot}00556 \text{ cm}$$

Let T be the difference of temperature.

Then $\quad T\times11{\cdot}2\times10^{-6}\times12=0{\cdot}00556$

$$\therefore \quad T=\mathbf{41{\cdot}4°C.}$$

17. A bronze liner of 6 cm external diameter is to be shrunk on a steel rod of 4·5 cm diameter. Calculate the maximum radial pressure between liner and rod if the maximum stress in the liner is limited to 1,200 kg/cm², also the difference between the liner bore and shaft diameter before shrinking.

E **for steel and bronze $2{\cdot}1\times10^{6}$ and 10^{6} kg/cm² respectively. Poisson's ratio 0·3 for both steel and bronze.** ((Lond. Univ.)

Bronze liner :

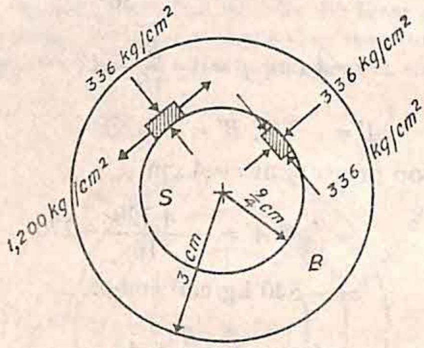

Fig. 18

At $\quad r=\frac{9}{4}$ cm, $f=1{,}200=\dfrac{16B}{81}+A \quad \ldots \quad (1)$

At $\quad r=3$ cm, $p=0=\dfrac{B}{9}-A \quad \ldots \quad (2)$

Solving $\quad A=432, \quad B=3{,}888$

At $\quad r=\frac{9}{4}$ cm, $p=\dfrac{16B}{81}-A$

$$=\frac{16\times3{,}888}{81}-432$$

$$=\mathbf{336\ kg/cm^2}$$

Steel rod :

$$f=-p=-336 \text{ kg/cm}^2\text{, both compressive}$$

Hoop strain in bronze at $r=\frac{9}{4}$ cm,

$$=\frac{1}{E_B}(1{,}200+0{\cdot}3\times 336)=\frac{1300{\cdot}8}{10^6}$$

$$=0{\cdot}0013008$$

Increase in internal diameter

$$=0{\cdot}0013008\times 4{\cdot}5=0{\cdot}00585 \text{ cm}$$

Hoop strain in steel at $r=\frac{9}{4}$ cm,

$$=\frac{1}{E_S}(-336+0{\cdot}3\times 336)=-\frac{235{\cdot}2}{2{\cdot}1\times 10^6}$$

$$=-0{\cdot}000112$$

Decrease in diameter

$$=0{\cdot}000112\times 4{\cdot}5=0{\cdot}000504 \text{ cm}$$

Difference in diameters

$$=0{\cdot}00585+0{\cdot}000504=\mathbf{0{\cdot}006354 \text{ cm.}}$$

18. A steel cylinder 16 cm external diameter and 12 cm internal diameter has another cylinder 20 cm external diameter shrunk on to it. If the maximum tensile stress induced in the outer cylinder is 800 kg/cm², find the radial compressive stress between the cylinders.

Determine the cicumferential stresses at inner and outer diameters of both cylinders and show, by means of a diagram, how these stresses vary with the radius. Calculate the necessary shrinkage allowance at the common surface. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

Outer Cylinder :

Maximum stress at $r=8$ cm,

$$800=\frac{B}{64}+A$$

External pressure $$0=\frac{B}{100}-A$$

Solving $$A=\frac{12{,}800}{41},\quad B=\frac{1{,}280{,}000}{41}$$

At $r=8$ cm, $$p=\frac{B}{64}-A$$

$$=\frac{1{,}280{,}000}{64\times 41}-\frac{12{,}800}{41}$$

$$=\mathbf{175{\cdot}6 \text{ kg/cm}^2}$$

$$r=10 \text{ cm},\ f=\frac{B}{100}+A$$

$$=\frac{1{,}280{,}000}{100\times 41}+\frac{12{,}800}{41}$$

$$=\mathbf{624\ kg/cm^2\ tension}$$

Hoop strain at the inner surface

$$=\frac{1}{E}\left(800+\frac{175{\cdot}6}{m}\right)$$

$$\text{Increase in diameter}=\frac{16}{E}\left(800+\frac{175{\cdot}6}{m}\right) \quad .. \quad (1)$$

Inner cylinder :

At $\quad r=8 \text{ cm},\ p=175{\cdot}6=\frac{B'}{64}-A'$

At $\quad r=6 \text{ cm},\ p=0=\frac{B'}{36}-A'$

Solving $\quad A'=-\frac{175{\cdot}6\times 16}{7},\ B'=-\frac{175{\cdot}6\times 576}{7}$

At $\quad r=6 \text{ cm},\ f=\frac{B'}{36}+A'$

$$=-\frac{175{\cdot}6\times 576}{36\times 7}-\frac{175{\cdot}6\times 16}{7}$$

$$=\mathbf{-803\ kg/cm^2\ comp.}$$

At $\quad r=8 \text{ cm},\ f=\frac{B'}{64}+A'$

$$=-\frac{175{\cdot}6\times 576}{64\times 7}-\frac{175{\cdot}6\times 16}{7}$$

$$=\mathbf{-627\ kg/cm^2\ comp.}$$

Hoop strain at the outer surface

$$=-\frac{1}{E}\left(627-\frac{175{\cdot}6}{m}\right)$$

$$\text{Decrease in diameter}=\frac{16}{E}\left(627-\frac{175{\cdot}6}{m}\right) \quad .. \quad (2)$$

∴ Shrinkage allowance on the common diameter

$$=(1)+(2)=\frac{16}{E}(800+627)$$

$$=\frac{16\times 1{,}427}{2\times 10^6}$$

$$=\mathbf{0{\cdot}01142\ cm}$$

The circumferential stresses and their variation are shown in Fig. 19.

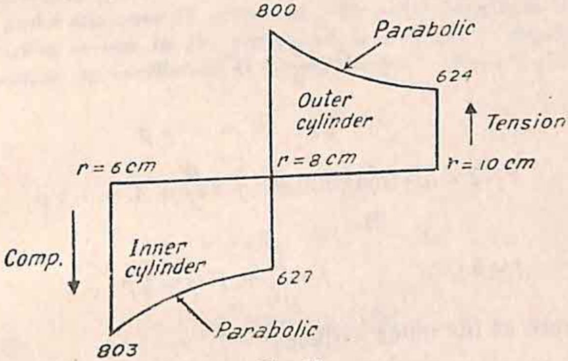

Fig. 19

19. A gun-metal cylinder, 12 cm external diameter and 7·994 cm internal diameter is forced on to a steel cylinder 8 cm external diameter and 4 cm internal diameter.

Calculate the maximum resulting stresses in the steel and gun-metal. Young's modulus to be taken 2×10^6 and 10^6 kg/cm² for steel and gun-metal respectively, and Poisson's ratio 0·35 for both metals. (Lond. Univ.)

Let p be the pressure at the common surface.

Gun-metal cylinder :

$$p=\frac{B}{16}-A$$

$$0=\frac{B}{36}-A$$

Solving $\qquad A=\frac{4}{5}p,\ B=\frac{144}{5}p$

At $\qquad r=4$ cm, $\quad f=\frac{B}{16}+A=\frac{144\,p}{16\times5}+\frac{4}{5}p$

$$=\tfrac{13}{5}\,p \text{ tension}$$

$$\text{Hoop strain} = \frac{1}{E_G}\left(\frac{13}{5}p + 0{\cdot}35\,p\right) = \frac{2{\cdot}95\,p}{E_G}$$

Increase in internal diameter

$$= \frac{2{\cdot}95\,p}{E_G} \times 8 = \frac{23{\cdot}6\,p}{E_G} \quad \ldots \quad (1)$$

Steel cylinder :

$$p = \frac{B'}{16} - A'$$

$$0 = \frac{B'}{4} - A'$$

Solving $A' = -\frac{4}{3}p, \quad B' = -\frac{16}{3}p$

At $r = 2$ cm, maximum $f = \frac{B'}{4} + A' = -\frac{8}{3}p$

At $r = 4$ cm, $f = \frac{B'}{16} + A' = -\frac{5}{3}p$

Hoop strain at the outer surface

$$= \frac{1}{E_S}\left(-\frac{5}{3}p + 0{\cdot}35\,p\right) = -\frac{3{\cdot}95\,p}{3E_S}$$

Decrease in the external diameter

$$= \frac{3{\cdot}95p}{3E_S} \times 8 = \frac{31{\cdot}6p}{3E_S} \quad \ldots \quad (2)$$

$\therefore$ Difference in diameters

$$= (1) + (2) = \frac{23{\cdot}6p}{E_G} + \frac{31{\cdot}6\,p}{3E_S}$$

i. e., $$0{\cdot}006 = \frac{23{\cdot}6p}{10^6} + \frac{31{\cdot}6p}{3 \times 2 \times 10^6}$$

$$\therefore \quad p = 208 \text{ kg/cm}^2$$

Maximum f in gun-metal at $r = 4$ cm,

$$= \frac{13}{5}p = \frac{13}{5} \times 208$$

$$= \mathbf{541 \text{ kg/cm}^2 \text{ tension}}$$

Maximum f in steel $= -\frac{8}{3}p = -\frac{8}{3} \times 208$

$$= \mathbf{-555 \text{ kg/cm}^2 \text{ comp.}}$$

20. A high tensile steel tyre, 2 cm thick, is shrunk on a cast-iron rim having 48 cm outside diameter and 6 cm thick. Find the inside diameter of the steel tyre to the nearest thousandth of a centimetre if, after shrinking on, the tyre exerts a radial pressure of 500 kg/cm² on the cast-iron rim.

E for steel$=2\cdot1\times10^6$ kg/cm²,
E for $C.\ I.=10^6$ kg/cm²,
$m=4$ for both. (Engineering Services, 1964)

(Ans. 47·774 cm)

21. A solid plug gauge of steel has a diameter of 2·0004 cm, and is forced into a ring gauge of the same material, which measures 2 cm inside diameter and 4 cm outside diameter. Its axial length is 2 cm. What is the maximum stress in the ring, and what force is required to slide the plug, assuming the coefficient of friction is 0·3 ? Take $E=2\times10^6$ kg/cm².

(Lond. Univ.)

Ring :

At $\quad r=1$ cm, $\quad p=B-A$

At $\quad r=2$ cm, $\quad 0=\frac{B}{4}-A$

Solving $\quad A=\frac{p}{3},\quad B=\frac{4p}{3}$

At $\quad r=1$ cm, maximum hoop stress

$$=B+A=\frac{4p}{3}+\frac{p}{3}$$

$$=\frac{5p}{3}\text{ tension}$$

Hoop strain $\quad =\frac{1}{E}\left(\frac{5p}{3}+\frac{p}{m}\right)=\frac{p}{E}\left(\frac{5}{3}+\frac{1}{m}\right)$

Increase in internal diameter

$$=\frac{2p}{E}\left(\frac{5}{3}+\frac{1}{m}\right) \quad \ldots \quad (1)$$

Plug :

$f=-p$, both compressive

Hoop strain $\quad =\frac{1}{E}\left(-p+\frac{p}{m}\right)=-\frac{p}{E}\left(1-\frac{1}{m}\right)$

Decrease in diameter $\quad =\frac{2p}{E}\left(1-\frac{1}{m}\right) \quad \ldots \quad (2)$

$\therefore$ Difference in diameters

$$=(1)+(2)=\frac{2p}{E}\left(\frac{5}{3}+1\right)$$

$$=\frac{16p}{3E}$$

$$\frac{16p}{3E}=0{\cdot}0004$$

$$\therefore \quad p=\frac{3\times 2\times 10^6\times 0{\cdot}0004}{16}=150 \text{ kg/cm}^2$$

Maximum f in ring $\quad =\frac{5p}{3}=\frac{5}{3}\times 150$

$=250$ kg/cm²

Area of friction surface $=\pi\times 2\times 2=4\pi$ cm²

Total radial pressure $P=150\times 4\pi=600\pi$ kg

Force $\quad F=0{\cdot}3\times 600\pi=565$ kg.

22. A steel cylindrical plug of 10 cm diameter is forced into a steel sleeve of 16 cm external diameter and 8 cm long. If the greatest circumferential stress in the sleeve is 900 kg/cm², find the torque required to turn the plug in the sleeve assuming the coefficient of friction between the plug and the sleeve is 0·2.

(Lond. Univ.)

(Ans. 99,100 kg cm)

23. A compound cylinder is to be made by shrinking an outer tube of 24 cm external diameter on to an inner tube of 12 cm internal diameter. Determine the common diameter at the junction if the greatest circumferential stress in the inner tube is to be two-thirds of the greatest circumferential stress in the outer tube.

(Lond. Univ.)

(Ans. 19·46 cm)

24. A cylinder consists of an outer tube of internal and external diameters 16 cm and 18 cm respectively shrunk on to an inner tube of internal and external diameters 8 cm and 16·01 cm respectively. The internal pressure developed in the cylinder is 3,000 kg/cm². If $E=2\times 10^6$ kg/cm², find the maximum hoop stress developed in each tube.

(Engineering Services, 1957)

Due to shrinkage :

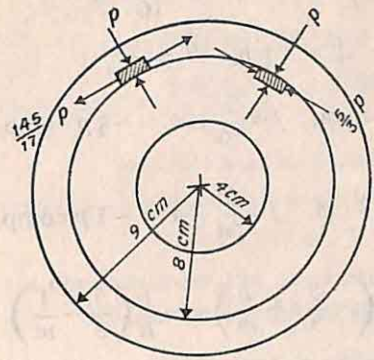

Fig. 20

Let p be the pressure at the common surface.

Outer tube :

At $\qquad r=8 \text{ cm}, \quad p=\dfrac{B}{64}-A$

At $\qquad r=9 \text{ cm}, \quad 0=\dfrac{B}{81}-A$

Solving $\qquad A=\frac{64}{17}p, \quad B=\dfrac{5,184}{17}p$

At $\qquad r=8 \text{ cm}, \quad f=\dfrac{B}{64}+A$

$$=\frac{5,184}{64\times 17}p+\frac{64}{17}p$$

$$=\tfrac{145}{17}p$$

Hoop Strain $=\dfrac{1}{E}\left(\dfrac{145}{17}p+\dfrac{p}{m}\right)=\dfrac{p}{E}\left(\dfrac{145}{17}+\dfrac{1}{m}\right)$

Increase in internal diameter

$$=\frac{16p}{E}\left(\frac{145}{17}+\frac{1}{m}\right) \tag{1}$$

Inner tube :

At $\qquad r=8 \text{ cm}, \quad p=\dfrac{B'}{64}-A'$

At $\quad r=4$ cm, $\quad 0=\dfrac{B'}{16}-A'$

Solving $\quad A'=-\frac{4}{3}p, \quad B'=-\frac{64}{3}p$

At $\quad r=4$ cm, $\quad f=\dfrac{B'}{16}+A'=-\frac{8}{3}p$ comp.

At $\quad r=8$ cm, $\quad f=\dfrac{B'}{64}+A'=-\frac{5}{3}p$ comp.

$$\text{Hoop strain}=\frac{1}{E}\left(-\frac{5}{3}p+\frac{p}{m}\right)=-\frac{p}{E}\left(\frac{5}{3}-\frac{1}{m}\right)$$

Decrease in external diameter

$$=\frac{16p}{E}\left(\frac{5}{3}-\frac{1}{m}\right)$$

Difference in diameters

$$=(1)+(2)=\frac{16p}{E}\left(\frac{145}{17}+\frac{5}{3}\right)$$

$$=\frac{16\times 520p}{51E}$$

i. e., $$\frac{16\times 520p}{51\times 2\times 10^6}=0{\cdot}01$$

$$\therefore \quad p=122{\cdot}6 \text{ kg/cm}^2$$

Outer tube, inner surface

$$f=\tfrac{145}{17}p=\tfrac{145}{17}\times 122{\cdot}6=1{,}046 \text{ kg/cm}^2 \text{ tension}$$

Inner tube, inner surface

$$f=-\tfrac{8}{3}p=-\tfrac{8}{3}\times 122{\cdot}6=-327 \text{ kg/cm}^2 \text{ comp.}$$

Due to internal pressure

At $\quad r=4$ cm, $\quad 3{,}000=\dfrac{B''}{16}-A''$

At $\quad r=9$ cm, $\quad 0=\dfrac{B''}{81}-A''$

Solving $\quad A''=\dfrac{9{,}600}{13}, \quad B''=\dfrac{9{,}600\times 81}{13}$

At $\quad r=4$ cm, $f=\dfrac{9{,}600\times 81}{16\times 13}+\dfrac{9{,}600}{13}$

$=4{,}480$ kg/cm² tension

At $\quad r=8$ cm, $f=\dfrac{9{,}600\times 81}{64\times 13}+\dfrac{9{,}600}{13}$

$=1{,}673$ kg/cm² tension

At $\quad r=8$ cm, maximum f in outer tube

$=1{,}046+1{,}673=$ **2,719 kg/cm² tension**

At $\quad r=4$ cm, maximum f in inner tube

$=-327+4{,}480=$ **4,153 kg/cm² tension.**

25. A steel plug 8 cm diameter is forced into a steel ring 14 cm external diameter and 6 cm wide. From a reading taken by fixing in a circumferential direction an electric-resistance strain gauge on the external surface of the ring, the strain is found to be $1{\cdot}49\times 10^{-4}$. Assuming $\mu=0{\cdot}2$ for the mating surface, find the force required to push the plug out of the ring. Also estimate the greatest hoop stress in the ring.

$E=2\times 10^6$ kg/cm².

Let p be the pressure at the common surface.

Steel ring :

At $\quad r=4$ cm, $\quad p=\dfrac{B}{16}-A$

At $\quad r=7$ cm, $\quad 0=\dfrac{B}{49}-A$

Solving $\quad A=\frac{16}{33}p,\quad B=\dfrac{16\times 49}{33}p$

At $\quad r=7$ cm, $\quad f=\dfrac{B}{49}+A$

$=\frac{16}{33}p+\frac{16}{33}p$

$=\frac{32}{33}p$

Hoop strain at the external surface

$$=\frac{32p}{33E}$$

i.e., $\quad \dfrac{32p}{33\times 2\times 10^6}=1{\cdot}49\times 10^{-4}$

$\therefore\quad p=307$ kg/cm²

Area of friction surface $= \pi \times 8 \times 6 = 48\pi$ cm²

Total radial pressure $P = 307 \times 48\pi$ kg

Force $F = 307 \times 48\pi \times 0{\cdot}2 = \mathbf{9{,}260\ kg}$

Greatest hoop stress at $r = 4$ cm,

$$= \frac{B}{16} + A = \tfrac{49}{33}p + \tfrac{16}{33}p = \tfrac{65}{33}p$$

$$= \tfrac{65}{33} \times 307$$

$$= \mathbf{605\ kg/cm^2}.$$

26. A steel rod, 6 cm diameter, is forced into a bronze casing having an outside diameter of 10 cm and thereby produces a hoop tension at the outer circumference of the casing of 350 kg/cm². Determine (*a*) the radial pressure between the rod and the casing, and (*b*) the rise in temperature which would just eliminate the force fit.

For steel, $E = 20 \times 10^5$ **kg/cm²,** $\frac{1}{m} = 0{\cdot}28$, $\alpha = 12 \times 10^{-6}$ **per degree C.**

For bronze, $E = 11 \times 10^5$ **kg/cm²,** $\frac{1}{m} = 0{\cdot}33$, $\alpha = 19 \times 10^{-6}$ **per degree C.**

(Lond. Univ.)

Bronze casing :

At $r = 5$ cm, $350 = \frac{B}{25} + A$

At $r = 5$ cm, $0 = \frac{B}{25} - A$

Solving $A = 175$, $B = 4{,}375$

At $r = 3$ cm, $p = \frac{B}{9} - A = \frac{4{,}375}{9} - 175$

$$= \mathbf{311\ kg/cm^2}$$

At $r = 3$ cm, $f = \frac{B}{9} + A = \frac{4{,}375}{9} + 175$

$$= 661 \text{ kg/cm}^2$$

Hoop strain at the inner surface

$$= \frac{1}{E_B}(661 + 0{\cdot}33 \times 311) = \frac{763}{11 \times 10^5}$$

$$= 0{\cdot}000694$$

Increase in internal diameter $= 6 \times 0{\cdot}000694$ cm (1)

Steel rod :

$$f = -p = -311 \text{ kg/cm}^2$$

$$\text{Hoop strain} = \frac{1}{E_s}\,(-311+0{\cdot}28\times 311) = -\frac{224}{20\times 10^5}$$

$$= -0{\cdot}000112$$

Decrease in 6 cm diameter $= 6\times 0{\cdot}000112$ (2)

$\therefore$ Difference in diameter $= (1)+(2) = 6\times 0{\cdot}000806$

Thus $(\alpha_B-\alpha_S)T\times 6 = 6\times 0{\cdot}000806$

or $7\times 10^{-6}\times T = 0{\cdot}000806$

$\therefore$ $T = 115$ degree C.

27. A steel shaft 3 cm diameter is to be encased in a bronze sleeve, 4·5 cm outside diameter, which is to be forced into position and, before forcing on, the inside diameter of the sleeve is 0·004 cm smaller than the diameter of the shaft. Find (*a*) the radial pressure between the shaft and sleeve, (*b*) the maximum hoop stress in the sleeve, (*c*) the change in outside diameter of the sleeve. *E* for steel $=20\times 10^5$ kg/cm², *E* for bronze $=12\times 10^5$ kg/cm², Poisson's ratio for steel $=0{\cdot}29$, Poisson's ratio for bronze $=0{\cdot}34$. (Lond. Univ.)

(Ans. 475 kg/cm²; 1,235 kg/cm²; 0·00285 cm)

28. A bronze bush having an outside diameter of 18 cm and an inside diameter of 10 cm is pressed into a recess in a body which is assumed to be perfectly rigid. If the diameter of the recess is 17·995 cm, find the radial pressure produced on the outer surface of the bush and the maximum hoop stress in the bush. Determine also the change in the inside diameter of the bush.

For bronze take $E=10^6$ kg/cm² and Poisson's ratio $=0{\cdot}35$. (Lond. Univ.)

Let p be the pressure on the outside surface of the bush.

At $r=9$ cm, $p=\dfrac{B}{81}-A$

At $r=5$ cm, $0=\dfrac{B}{25}-A$

Solving $A=-\dfrac{81}{56}p,\; B=-\dfrac{81\times 25}{56}p$

At $r=5$ cm, $f=\dfrac{B}{25}+A$

$$= -\tfrac{81}{56}p-\tfrac{81}{56}p = -\tfrac{81}{28}p$$

At $r=9$ cm, $f=\dfrac{B}{81}+A$

$$= -\tfrac{25}{56}p-\tfrac{81}{56}p = -\tfrac{53}{28}p$$

Hoop strain at the outer surface

$$=\frac{1}{E}\left(-\frac{53}{28}p+\frac{p}{m}\right)=-\frac{p}{E}\left(\frac{53}{28}-0{\cdot}35\right)$$

$$=-\frac{1{\cdot}543\,p}{E}$$

Decrease in the outer diameter $=\dfrac{1{\cdot}543\,p}{E}\times 18$

i. e., $$0{\cdot}005=\frac{1{\cdot}543\,p\times 18}{10^6}$$

$$\therefore\quad p=\mathbf{180\ kg/cm^2}$$

Maximum f at $r=5$ cm,

$$=-\tfrac{81}{28}\,p=-\tfrac{81}{28}\times 180=\mathbf{-521\ kg/cm^2\ comp.}$$

Hoop strain at the inner surface

$$=-\frac{521}{10^6}$$

Decrease in the internal diameter

$$=\frac{521}{10^6}\times 10=\mathbf{0{\cdot}00521\ cm.}$$

29. A tube 8 cm inside by 12 cm outside diameter is to be reinforced by shrinking on a second tube of 16 cm outside diameter. The compound tube is to withstand an internal pressure of 350 kg/cm² and the shrinkage allowance is to be such that the final maximum stress in each tube is to be the same. Calculate this stress and show on a diagram the variation of hoop stress in the two tubes. What is the initial difference of diameters before shrinking on ? $E=2\times 10^6$ kg/cm².

Due to shrinkage

Let p be the pressure at the common surface.

Outer tube :

At $\quad r=6$ cm, $p=\dfrac{B}{36}-A$

At $\quad r=8$ cm, $0=\dfrac{B}{64}-A$

Solving $\quad A=\frac{9}{7}\,p,\ B=\frac{576}{7}\,p$

At $\quad r=6$ cm, $f=\dfrac{B}{36}+A$

$$=\tfrac{16}{7}\,p+\tfrac{9}{7}\,p=\tfrac{25}{7}\,p$$

At $$r=8 \text{ cm}, \; f=\frac{B}{64}+A$$
$$=\tfrac{9}{7}p+\tfrac{9}{7}p=\tfrac{18}{7}p$$

Inner tube :

At $$r=6 \text{ cm}, \; p=\frac{B'}{36}-A'$$

At $$r=4 \text{ cm}, \; 0=\frac{B'}{16}-A'$$

Solving $$A'=-\tfrac{9}{5}p, \; B'=-\tfrac{144}{5}p$$

At $$r=4 \text{ cm}, \; f=\frac{B'}{16}+A'=-\frac{9}{5}p-\frac{9}{5}p=-\frac{18}{5}p$$

At $$r=6 \text{ cm}, \; f=\frac{B'}{36}+A'=-\frac{4}{5}p-\frac{9}{5}p=-\frac{13}{5}p$$

Due to internal pressure

At $$r=4 \text{ cm}, \; 350=\frac{B''}{16}-A''$$

At $$r=8 \text{ cm}, \quad 0=\frac{B''}{64}-A''$$

Solving $$A''=\frac{350}{3}, \; B''=\frac{22{,}400}{3}$$

At $$r=4 \text{ cm}, \; f=\frac{B''}{16}+A''=\frac{1{,}400}{3}+\frac{350}{3}=583 \text{ kg/cm}^2$$

At $$r=6 \text{ cm}, \; f=\frac{B''}{36}+A''=\frac{5{,}600}{27}+\frac{350}{3}=324 \text{ kg/cm}^2$$

At $$r=8 \text{ cm}, \; f=\frac{B''}{64}+A''=\frac{350}{3}+\frac{350}{3}=233 \text{ kg/cm}^2$$

Maximum hoop stress will occur at the inner surface of each tube. Equating these values for the two tubes we get

$$\tfrac{25}{7}p+324=-\tfrac{18}{5}p+583$$

or $$p=36{\cdot}1 \text{ kg/cm}^2$$

Maximum $$f=\tfrac{25}{7}p+324=\tfrac{25}{7}\times 36{\cdot}1+324$$
$$=453 \text{ kg/cm}^2$$

Stress diagram is shown in Fig. 21.

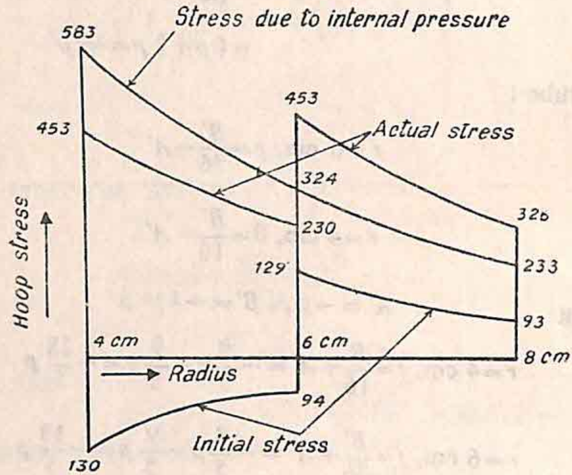

Fig. 21

Due to shrinkage

Hoop strain at the inner surface of outer tube

$$=\frac{1}{E}\left(129+\frac{36{\cdot}1}{m}\right)$$

Increase in the inner diameter $=\frac{12}{E}\left(129+\frac{36{\cdot}1}{m}\right)$

Hoop strain at the outer surface of inner tube

$$=\frac{1}{E}\left(-94+\frac{36{\cdot}1}{m}\right)=-\frac{1}{E}\left(94-\frac{36{\cdot}1}{m}\right)$$

Decrease in the outer diameter $=\frac{12}{E}\left(94-\frac{36{\cdot}1}{m}\right)$

Difference of diameter $=\frac{12}{E}\left(129+\frac{36{\cdot}1}{m}\right)+\frac{12}{E}\left(94-\frac{36{\cdot}1}{m}\right)$

$$=\frac{12}{E}\,(129+94)=\frac{12\times 223}{2\times 10^6}$$

$$=0{\cdot}001338 \text{ cm.}$$

30. A steel cylinder is shrunk on to another, the compound cylinder having an outside diameter of 16 cm, an inside diameter of 8 cm and a diameter of 12 cm at the contact surface. If the shrinkage produces a

radial pressure p_0 at the surfaces in contact, after which the compound cylinder is subjected to an internal pressure p_1, find the ratio of p_0 to p_1 so that the maximum hoop tensions in the two cylinders shall be the same.

(Lond. Univ.)

(Ans. 0·103)

31. A steel cylinder of outside diameter 24 cm and inside diameter 20 cm is shrunk on to one having diameters 20 cm and 16 cm, the interference fit being such that under an internal pressure p the inner tensile stress in both cylinders=850 kg/cm².

Find the initial difference in the nominal 20 cm diameters, and the value of p if $E=2\times 10^6$ kg/cm². (Lond. Univ.)

Due to shrinkage

Let p_0 be the pressure at the common surface.

Outer cylinder :

$$p_0=\frac{B}{100}-A$$

$$0=\frac{B}{144}-A$$

Solving $$A=\frac{25}{11}p_0,\ B=\frac{3{,}600}{11}p_0$$

At $$r=10 \text{ cm},\quad f=\frac{B}{100}+A=\frac{36}{11}p_0+\frac{25}{11}p_0$$

$$=\tfrac{61}{11}p_0 \quad \text{tension} \qquad \ldots (1)$$

Inner cylinder :

$$p_0=\frac{B'}{100}-A'$$

$$0=\frac{B'}{64}-A'$$

Solving $$A'=-\frac{25}{9}p_0,\ B'=-\frac{1{,}600}{9}p$$

At $$r=8 \text{ cm},\quad f=\frac{B'}{64}+A'=-\frac{25}{9}p_0-\frac{25}{9}p_0$$

$$=-\tfrac{50}{9}p_0 \quad \text{comp.} \qquad \ldots (2)$$

At $$r=10 \text{ cm},\quad f=\frac{B'}{100}+A'=-\frac{16}{9}p_0-\frac{25}{9}p_0$$

$$=-\tfrac{41}{9}p_0 \quad \text{comp.} \qquad \ldots (3)$$

Due to internal pressure

$$p=\frac{B''}{64}-A''$$

$$0=\frac{B''}{144}-A''$$

Solving $\quad A''=\frac{4}{5}p,\ B''=\frac{576}{5}p$

At $\quad r=8$ cm, $\quad f=\frac{B''}{64}+A''=\frac{9}{5}p+\frac{4}{5}p$

$$=\tfrac{13}{5}p \quad \text{tension} \qquad \ldots (4)$$

At $\quad r=10$ cm, $\quad f=\frac{B''}{100}+A''=\frac{5\cdot76}{5}p+\frac{4}{5}p$

$$=\frac{9\cdot76}{5}p \quad \text{tension} \qquad \ldots (5)$$

Total tensile stress at the inside of outer tube

$$=(1)+(5)=\frac{61}{11}p_0+\frac{9\cdot76}{5}p$$

i. e., $\quad \frac{61}{11}p_0+\frac{9\cdot76}{5}p=850$

or $\quad p_0+0\cdot352p=153\cdot3 \qquad \ldots (6)$

Total tensile stress at the inside of inner tube

$$=(2)+(4)=-\tfrac{50}{9}p_0+\tfrac{13}{5}p$$

i. e., $\quad -\frac{50}{9}p_0+\frac{13}{5}p=850$

or $\quad -p_0+0\cdot468p=153 \qquad (7)$

Solving equations 6 and 7

$$p=373\cdot5 \text{ kg/cm}^2, \quad p_0=21\cdot8 \text{ kg/cm}^2$$

Due to shrinkage :

Hoop strain at the inner surface of outer cylinder

$$=\frac{1}{E}\left(\frac{61}{11}p_0+\frac{p_0}{m}\right)$$

Increase in the internal diameter of outer cylinder

$$=\frac{20}{E}\left(\frac{61}{11}p_0+\frac{p_0}{m}\right) \qquad \ldots (8)$$

Hoop strain at the outer surface of inner cylinder

$$=-\frac{1}{E}\left(\frac{41}{9}p_0-\frac{p_0}{m}\right)$$

Decrease in the external diameter of inner cylinder

$$=\frac{20}{E}\left(\frac{41}{9}p_0-\frac{p_0}{m}\right) \quad .. \quad (9)$$

Initial difference in diameters

$$=(8)+(9)=\frac{20}{E}\left(\frac{61}{11}p_0+\frac{41}{9}p_0\right)$$

$$=\frac{20}{E}\times 10{\cdot}1\,p_0=\frac{20\times 10{\cdot}1\times 21{\cdot}8}{2\times 10^6}$$

$$=0{\cdot}0022 \text{ cm.}$$

32. The external diameter of a steel hub is 16 cm and the internal diameter increases 0·01 cm when shrunk on to a solid steel shaft of 10 cm diameter. Find the reduction in diameter of the shaft, the radial pressure between the hub and the shaft and the hoop stress at the inner surface of the hub. Poisson's ratio = 0·304, $E=2\times 10^6$ kg/cm². (Lond. Univ.)

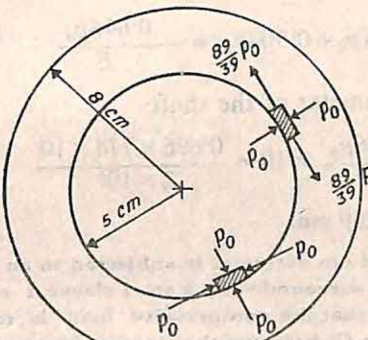

Fig. 22

Let p_0 be the pressure at the common surface.

Steel hub :

$$p_0=\frac{B}{25}-A \quad .. \quad (1)$$

$$0=\frac{B}{64}-A \quad .. \quad (2)$$

Solving equations 1 and 2

$$A=\frac{25}{39}p_0,\; B=\frac{1{,}600}{39}p_0$$

At $\quad r=5 \text{ cm}, f=\frac{B}{25}+A=\frac{89}{39}p_0$

Hoop strain at the inner surface

$$=\frac{1}{E}\left(\frac{89}{39}p_0+0{\cdot}304p_0\right)$$

$$=\frac{2{\cdot}584p_0}{E}$$

Increase in internal diameter

$$=\frac{2{\cdot}584p_0}{E}\times 10 \text{ cm}$$

$$\therefore \quad \frac{2{\cdot}584p_0}{2\times 10^6}\times 10=0{\cdot}01$$

or $\qquad p_0=\mathbf{774\ kg/cm^2}$

At $\qquad r=5$ cm, $f=\frac{89}{39}p_0=\mathbf{1{,}766\ kg/cm^2}$

Hoop strain at the outer surface of the shaft

$$=\frac{1}{E}(-p_0+0{\cdot}304p_0)=-\frac{0{\cdot}696p_0}{E}$$

Decrease in the diameter of the shaft

$$=\frac{0{\cdot}696p_0}{E}\times 10=\frac{0{\cdot}696\times 774\times 10}{2\times 10^6}$$

$$=\mathbf{0{\cdot}00269\ cm.}$$

33. A short steel rod 4 cm diameter is subjected to an axial compressive load of 20 tonnes. It is surrounded by a steel sleeve 1 cm thick, slightly shorter than the rod, so that the compressive load is taken only by the rod. Assuming a close fit between the sleeve and the rod before the load is applied, and neglecting friction, find: (*a*) the pressure between the sleeve and the rod, (*b*) the maximum tension in the sleeve. Take Poisson's ratio = 0·3.

(Lond. Univ,)

Steel rod :

$$\text{Longitudinal stress } f_l=\frac{20{,}000}{\frac{\pi}{4}\times 4^2}=1{,}592 \text{ kg/cm}^2 \text{ comp.}$$

Let p be the pressure at the common surface.

$f=-p$, both compressive

Hoop strain at the outer surface

$$=\frac{1}{E}(-p+0{\cdot}3p+0{\cdot}3\times 1{,}592)$$

$$=\frac{1}{E}(-0{\cdot}7p+477\ 6)$$

Steel sleeve :

At $$r=2 \text{ cm},\ p=\frac{B}{4}-A$$

At $$r=3 \text{ cm},\ 0=\frac{B}{9}-A$$

Solving $$A=\frac{4}{5}\,p,\ B=\frac{36}{5}\,p$$

At $$r=2 \text{ cm},\ f=\frac{B}{4}+A=\frac{13}{5}\,p$$

Hoop strain at the inner surface

$$=\frac{1}{E}\left(\frac{13}{5}p+0{\cdot}3p\right)=\frac{2{\cdot}9p}{E}$$

Hoop strain in the rod and the sleeve will be equal at the common surface.

$$\therefore\quad \frac{1}{E}(-0{\cdot}7p+477{\cdot}6)=\frac{2{\cdot}9p}{E}$$

$$\therefore\quad p=\mathbf{132{\cdot}7}\ \textbf{kg/cm}^2$$

Maximum tension in sleeve

$$=\tfrac{13}{5}\,p=345 \text{ kg/cm}^2.$$

34. A steel plunger for a hydraulic pump is 4 cm diameter and is encased in a bronze casing 1 cm thick. Assuming a perfect fit, without initial stress, calculate the maximum stresses occurring in the bronze and steel when under a radial pressure of 500 kg/cm². *E* for bronze and steel, 8×10^5 and 20×10^5 kg/cm² respectively. Poisson's ratio for bronze 0·33 and for steel 0·3. (Lond. Univ.)

Let p_0 be the pressure at the common surface

Steel plunger :

$$f=-p_0 \text{ comp.}$$

Hoop strain at the outer surface

$$=\frac{1}{E_s}(-p_0+0{\cdot}3p_0)=-\frac{0{\cdot}7p_0}{E_s}$$

Bronze casing :

At $$r=2 \text{ cm},\ p_0=\frac{B}{4}-A$$

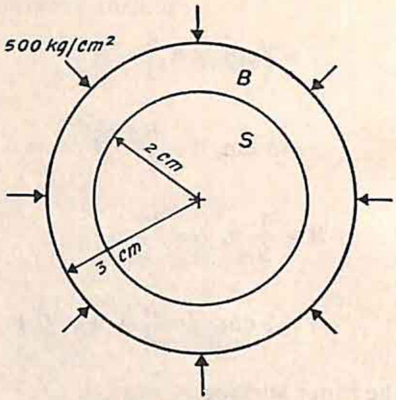

Fig. 23

At $$r = 3 \text{ cm},\ 500 = \frac{B}{9} - A$$

Solving $$A = \frac{4}{5}\,p_0 - 900,\ B = \frac{36}{5}\,p_0 - 3{,}600$$

At $$r = 2 \text{ cm},\ f = \frac{B}{4} + A = \frac{13}{5}\,p_0 - 1{,}800$$

Hoop strain at the inner surface

$$= \frac{1}{E_B}\left(\frac{13}{5}\,p_c - 1{,}800 + 0{\cdot}33p_0\right)$$

$$= \frac{1}{E_B}(2{\cdot}93p_0 - 1{,}800)$$

Hoop strain in steel plunger and bronze casing will be equal at the common surface.

$$\therefore\ -\frac{0{\cdot}7p_0}{20 \times 10^5} = \frac{1}{8 \times 10^5}\,(2{\cdot}93\,p_0 - 1{,}800)$$

Solving $$p_0 = 561 \text{ kg/cm}^2$$

Stresses in bronze :

At $$r = 2 \text{ cm},\ f = \frac{13}{5}\,p_0 - 1{,}800$$
$$= -341 \text{ kg/cm}^2 \text{ comp.}$$

At $$r = 3 \text{ cm},\ f = \frac{B}{9} + A = \frac{8}{5}\,p_0 - 1{,}300$$

$$= -402 \text{ kg/cm}^2 \text{ comp.}$$

Hence maximum hoop stress = **402 kg/cm² comp.**

Maximum radial stress = **561 kg/cm² comp.**

Stresses in steel :

$$f=-p_0=-561 \text{ kg/cm}^2 \text{ both compressive.}$$

35. A bronze cylinder of 30 cm internal diameter and 40 cm external diameter is surrounded by a closely fitting steel sleeve of 45 cm external diameter. Calculate the maximum hoop stresses in the steel and bronze when an internal pressure of 300 kg/cm² is applied to the compound cylinder, assuming that before the application of this pressure, the contact stress at the common surface is zero.

For steel, $E=2\times10^6$ kg/cm², Poisson's ratio = 0·28.

For bronze, $E=10^6$ kg/cm², Poisson's ratio = 0·35.

(Lond. Univ.)

(Ans. 863 kg/cm² tension, 609 kg/cm² tension.)

36. A steel bar of 4 cm diameter is pressed into a steel sleeve so that when assembled the radial pressure is 150 kg/cm² and the circumferential stress at the inside of the sleeve is 240 kg/cm². Determine the increase of radial pressure when the bar is subjected to an axial compressive load of 5,000 kg. Poisson's ratio = 0·304. (Lond. Univ.)

At the inner surface of the sleeve $\quad f=\dfrac{R_1^2+R_2^2}{R_1^2-R_2^2}.p$

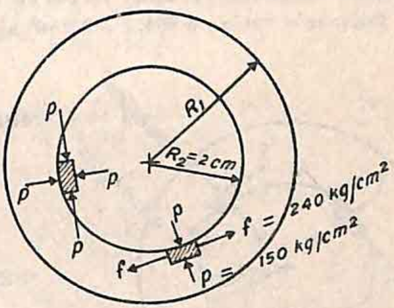

Fig. 24

Initially $f=240$ kg/cm²,

$p=150$ kg/cm²

$$\therefore \quad \frac{R_1^2+R_2^2}{R_1^2-R_2^2}=\frac{f}{p}$$

$$=\tfrac{240}{150}=1{\cdot}6$$

Finally when the axial load of 5,000 kg is applied on the bar

Bar :

$$f_l=\frac{5{,}000}{\frac{\pi}{4}\times4^2}=398 \text{ kg/cm}^2 \text{ compression}$$

Let δp be the increase in radial pressure.

Increase in hoop stress $=\delta p$

Change in hoop strain $=\dfrac{1}{E}\left(-\delta p+\dfrac{\delta p}{m}+\dfrac{f_l}{m}\right)$

Sleeve :

Increase in hoop stress, $\delta f=\dfrac{R_1^2+R_2^2}{R_1^2-R_2^2}.\delta p=1{\cdot}6\delta p$

$$\text{Change in hoop strain} = \frac{1}{E}\left(1\cdot6\delta p + \frac{\delta p}{m}\right)$$

Change in hoop strain at the common surface will be equal.

$$\therefore \quad \frac{1}{E}\left(-\delta p + \frac{\delta p}{m} + \frac{f_l}{m}\right) = \frac{1}{E}\left(1\cdot6\delta p + \frac{\delta p}{m}\right)$$

or $$2\cdot6\delta p = \frac{f_l}{m}$$

$$\therefore \quad \delta p = \frac{0\cdot304 \times 398}{2\cdot6} = \mathbf{46\cdot5\ kg/cm^2}.$$

37. A solid steel shaft of 2 cm diameter is pressed into a steel sleeve. Find the initial difference of diameters when the common radial pressure is 100 kg/cm² and the circumferential stress is 200 kg/cm² at the inside of the sleeve.

Find the axial compressive load that should be applied to the shaft to increase the radial pressure to 140 kg/cm².

Poisson's ratio = 0·304, $E = 2 \times 10^6$ kg/cm².

(Lond. Univ.)

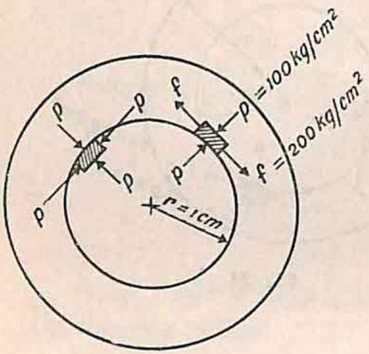

Fig. 25

(a) Hoop strain at the inside of the sleeve

$$= \frac{1}{E}\left(f + \frac{p}{m}\right)$$

$\therefore$ Increase in the internal diameter of the sleeve

$$= \frac{2}{E}\left(f + \frac{p}{m}\right) \quad \cdots \quad (1)$$

Hoop strain at the outside of the shaft

$$= \frac{1}{E}\left(-p + \frac{p}{m}\right) = -\frac{1}{E}\left(p - \frac{p}{m}\right)$$

$\therefore$ Decrease in the diameter of the shaft

$$= \frac{2}{E}\left(p - \frac{p}{m}\right) \quad \cdots \quad (2)$$

$\therefore$ Initial difference of diameters

$$= (1) + (2) = \frac{2}{E}(f + p) = \frac{2}{E} \times 300$$

$$= \mathbf{0\cdot0003\ cm.}$$

(b) Longitudinal stress

$$f_l = \frac{P}{\frac{\pi}{4}\times 2^2} = \frac{P}{\pi} \text{ kg/cm}^2 \text{ comp.}$$

Sleeve :

Let f' and p' be the altered stresses.

Then $$\frac{f'}{p'} = \frac{200}{100} = 2$$

$$\therefore \quad f' = 2p' = 280 \text{ kg/cm}^2$$

Hoop strain at the inside $= \frac{1}{E}\left(f' + \frac{p'}{m}\right)$

$\therefore$ Increase in internal diameter $= \frac{2}{E}\left(f' + \frac{p'}{m}\right)$ .. (3)

Shaft :

Hoop strain at the outside $= -\frac{1}{E}\left(p' - \frac{p'}{m} - \frac{f_l}{m}\right)$

$\therefore$ Decrease in diameter $= \frac{2}{E}\left(p' - \frac{p'}{m} - \frac{f_l}{m}\right)$ .. (4)

$\therefore$ Difference in diameters

$$= (3) + (4) = \frac{2}{E}\left(f' + p' - \frac{f_l}{m}\right)$$

$$= \frac{2}{E}\left(280 + 140 - \frac{0{\cdot}304P}{\pi}\right)$$

$$= \frac{2}{E} \times 300 \text{ (original difference)}$$

$$\therefore \quad P = \mathbf{1{,}240 \text{ kg.}}$$

38. A steel cylinder 5 cm bore and 8 cm outside diameter is fitted with 'onze liner of 4 cm bore. Calculate the greatest stress produced in both e cylinder and liner when the temperature is increased 100°C.
Coefficient of expansion: steel, $11{\cdot}6\times10^{-6}$; bronze, $18{\cdot}9\times10^{-6}$ per °C.
Modulus of elasticity: steel, 2×10^6; bronze, 10^6 kg/cm².
Poisson's ratio for both steel and bronze 0·29. (Lond. Univ.)

Let p be the pressure at the common surface.

Steel cylinder :

At $$r = \frac{5}{2} \text{ cm}, \quad p = \frac{4B}{25} - A$$

At $\quad r=4 \text{ cm}, \quad 0=\frac{B}{16}-A$

Solving $\quad A=\frac{25}{39}p, \quad B=\frac{400}{39}p$

At $\quad r=\frac{5}{2} \text{ cm}, \quad f=\frac{4B}{25}+A=\frac{89}{39}p$

Hoop strain, $\quad e_S=\frac{1}{E_S}\left(\frac{89}{39}p+0{\cdot}29p\right)=\frac{2{\cdot}57p}{E_S}$

Bronze liner :

At $\quad r=\frac{5}{2} \text{ cm}, \; p=\frac{4B'}{25}-A'$

At $\quad r=2 \text{ cm}, \; 0=\frac{B'}{4}-A'$

Solving $\quad A'=-\frac{25}{9}p, \; B'=-\frac{100}{9}p$

At $\quad r=\frac{5}{2} \text{ cm}, \; f=\frac{4B'}{25}+A'=-\frac{41}{9}p$

Hoop strain, $\quad e_B=\frac{1}{E_B}\left(-\frac{41}{9}p+0{\cdot}29p\right)=-\frac{4{\cdot}27p}{E_B}$

We know that $\frac{2{\cdot}57p}{E_S}+\frac{4{\cdot}27p}{E_B}=T(\alpha_B-\alpha_S)$

or

$$\frac{2{\cdot}57p}{2\times10^6}+\frac{4{\cdot}27p}{10^6}=100(18{\cdot}9\times10^{-6}-11{\cdot}6\times10^{-6})$$

or

$$p=131{\cdot}4 \text{ kg/cm}^2$$

Greatest hoop stress in steel cylinder

$$=\frac{89}{39}p=300 \text{ kg/cm}^2 \text{ tension}$$

Greatest hoop stress in bronze liner at $r=2$ cm,

$$=\frac{B'}{4}+A'=-\frac{50}{9}p=-730 \text{ kg/cm}^2 \text{ comp.}$$

39. A brass rod is a firm fit inside a steel tube 2 cm inside diameter and 4 cm outside diameter when the materials are at 15°C. If the rod and tube are now heated to a temperature of 150°C, calculate the maximum stresses in the brass and steel. Take the coefficient of expansion for steel and brass as $10{\cdot}8\times10^{-6}$ and 18×10^{-6} per degree C respectively,

Young's modulus as 20×10^5 and $8 \cdot 5 \times 10^5$ kg/cm² for steel and brass respectively and Poisson's ratio $\frac{1}{3}$ in both cases. Neglect longitudinal frictional forces between the rod and tube. (Engineering Services, 1968)

(Ans. 545 kg/cm² comp., 908 kg/cm² tension.)

40. Find the thickness of a spherical shell of 8 cm internal diameter, to withstand an internal pressure of 300 kg/cm², if the permissible tensile stress is 600 kg/cm², and shear stress 450 kg/cm².

What is the change of thickness under pressure of such a shell ? $E=2\times 10^6$ kg/cm², Poisson's ratio$=0\cdot 3$.

$$p \text{ compression} = \frac{B}{r^3} - A$$

$$f \text{ tension} = \frac{B}{2r^3} + A$$

Let R_1 be the external radius of the shell.

At $\quad r=4 \text{ cm}, \; 300 = \frac{B}{64} - A$

At $\quad r=R_1, \quad 0 = \frac{B}{R_1^3} - A$

Solving $\quad A = \frac{19{,}200}{R_1^3 - 64}, \; B = \frac{19{,}200 R_1^3}{R_1^3 - 64}$

$$f = \frac{B}{2r^3} + A = \frac{19{,}200 R_1^3}{2r^3(R_1^3 - 64)} + \frac{19{,}200}{R_1^3 - 64}$$

$$= \frac{19{,}200}{R_1^3 - 64}\left(\frac{R_1^3}{2r^3} + 1\right)$$

Maximum hoop stress at $r=4$ cm,

$$f_2 = \frac{19{,}200}{R_1^3 - 64}\left(\frac{R_1^3}{128} + 1\right) = \frac{150(R_1^3 + 128)}{R_1^3 - 64}$$

$$\therefore \quad 600 = \frac{150(R_1^3 + 128)}{R_1^3 - 64}$$

or $\quad R_1^3 = 128$

$$\therefore \quad R_1 = 5 \cdot 04 \text{ cm}$$

Maximum shear stress,

$$q = \tfrac{1}{2}(f_2 + 300) = \tfrac{1}{2}\left[\frac{150(R_1^3 + 128)}{R_1^3 - 64} + 300\right]$$

$$= \frac{225 R_1^3}{R_1^3 - 64}$$

$$\therefore \quad 450 = \frac{225 R_1^3}{R_1^3 - 64}$$

or $$R_1^3 = 128$$

$$\therefore \quad R_1 = 5{\cdot}04 \text{ cm; same as before.}$$

Thickness $= 5{\cdot}04 - 4 =$ **1·04 cm**

Hoop strain at the inner surface

$$= \frac{1}{E}(600 - 0{\cdot}3 \times 600 + 0{\cdot}3 \times 300) = \frac{510}{E}$$

Increase in internal radius

$$= \frac{510}{2 \times 10^6} \times 4 = 0{\cdot}00102 \text{ cm}$$

Hoop stress at the outer surface

$$= \frac{19{,}200}{R_1^3 - 64} \times \frac{3}{2} = \frac{28{,}800}{R_1^3 - 64} = \frac{28{,}800}{128 - 64}$$

$$= 450 \text{ kg/cm}^2$$

Hoop strain at the outer surface

$$= \frac{1}{E}(450 - 0{\cdot}3 \times 450) = \frac{315}{E}$$

Increase in external radius

$$= \frac{315}{2 \times 10^6} \times 5{\cdot}04 = 0{\cdot}000794 \text{ cm}$$

Decrease in thickness

$$= 0{\cdot}00102 - 0{\cdot}000794 = \mathbf{0{\cdot}000226 \text{ cm.}}$$

CHAPTER V

ROTATING RINGS, DISCS AND CYLINDERS

1. The thin rim of a wheel 80 cm diameter is made of steel, weighing 7·8 g/cm³ Neglecting the effect of the spokes, how many revolutions per minute may it make without the stress exceeding 1,500 kg/cm², and how much is the diameter of the wheel increased ? $E=2{\cdot}1\times10^6$ kg/cm².

$$f=\frac{\rho v^2}{g}$$

$$1{,}500=\frac{7{\cdot}8\times v^2}{1{,}000\times 981}$$

or
$$v^2=\frac{1{,}500\times 1{,}000\times 981}{7{\cdot}8}$$

$$\therefore \quad v=13{,}740 \text{ cm/sec.}$$

But
$$v=\frac{\pi DN}{60}$$

or
$$13{,}740=\frac{\pi\times 80\times N}{60}$$

$$\therefore \quad N=\mathbf{3{,}280\ r.\ p.\ m.}$$

Hoop strain
$$=\frac{f}{E}=\frac{1{,}500}{2{\cdot}1\times 10^6}$$

$$\therefore \quad \text{Increase in diameter}=\frac{1{,}500\times 80}{2{\cdot}1\times 10^6}=\mathbf{0{\cdot}0571\ cm.}$$

2. A disc of 24 cm diameter has a central hole of 4 cm diameter and runs at 4,000 r. p. m. Calculate the hoop stress at the inner and outer radius given that $m=4$ and $\rho=7{\cdot}8$ g/cm³.

At the inside

$$f=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times 4{,}000}{60}\right)^2}{1{,}000\times 4\times 981}\times(3{\cdot}25\times 12^2+0{\cdot}75\times 2^2)$$

$$=\mathbf{164{\cdot}3\ kg/cm^2}$$

At the outside

$$f=\frac{\rho\omega^2}{4g}\left\{\left(1-\frac{1}{m}\right)R_1^2+\left(3+\frac{1}{m}\right)R_2^2\right\}$$

$$=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times4{,}000}{60}\right)^2}{1{,}000\times4\times981}(0{\cdot}75\times12^2+3{\cdot}25\times2^2)$$

$$=\mathbf{42{\cdot}2\ kg/cm^2}.$$

3. A circular saw 5 mm thick, 90 cm diameter, is secured upon a 10 cm shaft. The steel of which the saw is composed has a weight of 8·1 g/cm³, and $m=3{\cdot}5$. Determine the permissible speed if the allowable hoop stress is 2,500 kg/cm², and find the maximum value of the radial stress.

$$f=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$2{,}500=\frac{8{\cdot}1\times\omega^2}{1{,}000\times4\times981}\left\{\left(3+\frac{1}{3{\cdot}5}\right)45^2+\left(1-\frac{1}{3{\cdot}5}\right)5^2\right\}$$

$$=\frac{8{\cdot}1\times\omega^2\times6{,}670}{1{,}000\times4\times981}$$

or $$\omega^2=\frac{2{,}500\times1{,}000\times4\times981}{8{\cdot}1\times6{,}670}=181{,}600$$

$$\therefore\quad \omega=426\text{ rad/sec.}$$

$$N=\frac{60\omega}{2\pi}=\frac{60\times426}{2\pi}=\mathbf{4{,}070\ r.p.m.}$$

Maximum $$p=\left(3+\frac{1}{m}\right)\frac{\rho\omega^2}{8g}(R_1-R_2)^2$$

$$=\left(3+\frac{1}{3{\cdot}5}\right)\times\frac{8{\cdot}1\times181{,}600\times40^2}{1{,}000\times8\times981}$$

$$=\mathbf{985\ kg/cm^2}.$$

4. A disc of uniform thickness is 80 cm in diameter and has a pin hole at the centre. Determine the maximum hoop stress in the disc when it rotates at 3,000 r.p.m. Density of disc material is 7·8 g/cm³ and Poisson's ratio = 0·304.

For a hollow disc

maximum $$f=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

When R_2 is very small

maximum $$f=\frac{\rho\omega^2}{4g}\left(3+\frac{1}{m}\right)R_1^2$$

$$=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times 3{,}000}{60}\right)^2}{1{,}000\times 4\times 981}\times 3{\cdot}304\times 40^2$$

$$=1{,}037\ \text{kg/cm}^2$$

5. A thin disc, 80 cm diameter, is having a central hole of 20 cm diameter. Find the maximum hoop stress if the maximum radial stress is 200 kg/cm². Assume $m=4$.

$$p=\left(3+\frac{1}{m}\right)\frac{\rho\omega^2}{8g}(R_1-R_2)^2$$

or $$200=\left(3+\frac{1}{4}\right)\times\frac{\rho\omega^2}{8g}\times 30^2$$

$$\therefore\quad \frac{\rho\omega^2}{g}=\frac{64}{117}$$

$$f=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$=\frac{64}{4\times 117}\left\{\left(3+\frac{1}{4}\right)0^2+\left(1-\frac{1}{4}\right)\times 10^2\right\}$$

$$=721\ \text{kg/cm}^2.$$

6. The rotor of a De Laval steam turbine is 60 cm diameter at the blade ring and is 9·5 cm thick at the centre. Calculate the thickness at the blade ring if the r.p.m. are 10,000, the stress uniform and equal to 1,800 kg/cm² and the density of the material is 7·8 g/cm³.

Thickness $$t=t_0\,e^{\frac{-\rho r^2\omega^2}{2fg}}$$

At $r=30$ cm, $$\frac{\rho r^2\omega^2}{2fg}=\frac{7{\cdot}8\times 30^2\times\left(\dfrac{2\pi\times 10{,}000}{60}\right)^2}{1{,}000\times 2\times 1{,}800\times 981}=2{\cdot}18$$

$$\therefore\quad t=9{\cdot}5\times e^{-2\cdot 18}=9{\cdot}5\times 0{\cdot}1131$$
$$=1{\cdot}074\ \text{cm}.$$

7. A De Laval steam turbine rotor is 16 cm diameter below the blade ring, and 6 mm thick, the running speed being 30,000 r.p.m. If the material weighs 7·8 g/cm³, and the allowable stress is 1,500 kg/cm², what is the thickness of the rotor at a radius of 4 cm and at the centre? Assume uniform strength.

$$t = t_0 e^{\frac{-\rho r^2 \omega^2}{2fg}}$$

$$\frac{\rho\omega^2}{2fg} = \frac{7{\cdot}8 \times \left(\dfrac{2\pi \times 30{,}000}{60}\right)^2}{1{,}000 \times 2 \times 1{,}500 \times 981} = 0{\cdot}0262$$

At $r = 8$ cm, $0{\cdot}6 = t_0 \times e^{-0{\cdot}0262 \times 64}$

$$= t_0 \times e^{-1{\cdot}677}$$

$$\therefore \quad t_0 = 0{\cdot}6 \times e^{1{\cdot}677} = 0{\cdot}6 \times 5{\cdot}35$$

$$= \mathbf{3{\cdot}21\ cm.}$$

At $r = 4$ cm, $t = 3{\cdot}21 \times e^{-0{\cdot}0262 \times 16} = 3{\cdot}21 \times e^{-0{\cdot}419}$

$$= 3{\cdot}21 \times 0{\cdot}658$$

$$= \mathbf{2{\cdot}11\ cm.}$$

8. A steel turbine disc is to be designed so that between radii of 25 cm and 40 cm the radial and circumferential stresses are to be constant and both equal to 600 kg/cm², when running at 3,000 r.p.m. If the axial thickness is 12 mm at the outer age of this zone, what should it be at the inner edge ? Assume $\rho = 7{\cdot}8$ g/cm³.

$$t = Ae^{\frac{-\rho r^2 \omega^2}{2fg}}$$

At $r = 40$ cm, $\quad 1{\cdot}2 = Ae^{\frac{-\rho\omega^2}{2fg} \times 1{,}600}$

$$\therefore \quad A = 1{\cdot}2 e^{\frac{\rho\omega^2}{2fg} \times 1{,}600}$$

At $r = 25$ cm, $\quad t = Ae^{\frac{-\rho\omega^2}{2fg} \times 625}$

$$= 1{\cdot}2 e^{\frac{\rho\omega^2}{2fg} \times 1{,}600} \times e^{\frac{-\rho\omega^2}{2fg} \times 625}$$

$$= 1{\cdot}2 \times e^{\frac{\rho\omega^2}{2fg} \times 975}$$

$$= 1{\cdot}2 \times e^{\dfrac{7{\cdot}8 \times \left(\dfrac{2\pi \times 3{,}000}{60}\right)^2 \times 975}{1{,}000 \times 2 \times 600 \times 981}}$$

$$= 1{\cdot}2 \times e^{0{\cdot}638} = 1{\cdot}2 \times 1{\cdot}892$$

$$= 2{\cdot}27 \text{ cm.}$$

9. Find the maximum hoop stress in a solid cylinder of cast iron 40 cm diameter when making 2,000 rotations per minute about its axis, taking the weight 7·2 g/cm³ and Poisson's ratio $\frac{1}{3}$.

Maximum f at the centre

$$= \frac{3m-2}{m-1} \cdot \frac{\rho\omega^2}{8g} R^2$$

$$= \frac{7}{2} \times \frac{7{\cdot}2 \times \left(\dfrac{2\pi \times 2{,}000}{60}\right)^2 \times 20^2}{1{,}000 \times 8 \times 981}$$

$$= 56{\cdot}3 \text{ kg/cm}^2.$$

10. Calculate the maximum stress in a long cylinder 4 cm inside diameter and 20 cm outside diameter rotating at 3,000 r.p.m. $\rho = 7{\cdot}8$ g/cm³; $m = \frac{10}{3}$.

Maximum f at the inside

$$= \frac{\rho\omega^2}{4g(m-1)}\{(3m-2)R_1^2 + (m-2)R_2^2\}$$

$$= \frac{7{\cdot}8 \times \left(\dfrac{2\pi \times 3{,}000}{60}\right)^2}{1{,}000 \times 4 \times 981 \times \frac{7}{3}} \times \{8 \times 10^2 + \tfrac{4}{3} \times 2^2\}$$

$$= 67{\cdot}7 \text{ kg/cm}^2.$$

11. Compare the periphery velocities for the same maximum intensity of stress of (a) a solid cylinder, (b) a solid thin disc, (c) a thin ring. Take the velocity of the ring as unity and $m = 3{\cdot}5$.

Solid cylinder :

$$f = \frac{3m-2}{m-1} \cdot \frac{\rho\omega^2 R^2}{8g}$$

$$= \frac{8{\cdot}5}{2{\cdot}5} \times \frac{\rho v_1^2}{8g}$$

$$\therefore \quad v_1^2 = 2{\cdot}35 \times \frac{gf}{\rho}.$$

Solid thin disc :

$$f=\left(3+\frac{1}{m}\right)\frac{\rho\omega^2R^2}{8g}=\left(3+\frac{1}{3\cdot5}\right)\times\frac{\rho v_2^2}{8g}$$

$$\therefore\quad v_2^2=2\cdot43\times\frac{gf}{\rho}$$

Thin ring :

$$f=\frac{\rho v_3^2}{g}$$

$$\therefore\quad v_3^2=\frac{gf}{\rho}$$

Hence $v_1^2 : v_2^2 : v_3^2=2\cdot35 : 2\cdot43 : 1$

or $v_1 : v_2 : v_3=\mathbf{1\cdot533 : 1\cdot559 : 1}$.

12. A circular disc 50 cm outside diameter has a central hole and rotates at a uniform speed about an axis through its centre. The diameter of the hole is such that the maximum stress due to rotation is 85% of that in a thin ring whose mean diameter is also 50 cm. If both are of the same material and rotate at the same speed determine (a) the diameter of the central hole, (b) the speed of rotation, if the allowable stress in the disc is 900 kg/cm². $\rho=7\cdot8$ g/cm³, and $m=\frac{10}{3}$.

Thin ring :

$$f=\frac{\rho r^2\omega^2}{g}$$

$$\frac{900}{0\cdot85}=\frac{7\cdot8\times25^2\times\omega^2}{1{,}000\times981}$$

$$\therefore\quad \omega^2=213{,}000$$

$$\omega=462 \text{ rad/sec.}$$

$$N=\frac{60\omega}{2\pi}=\frac{60\times462}{2\pi}=\mathbf{4{,}410 \text{ r. p. m.}}$$

Hollow disc :

Maximum f at the inside

$$=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$900 = \frac{7{\cdot}8 \times 213{,}000}{1{,}000 \times 4 \times 981} \times (3{\cdot}3 \times 25^2 + 0{\cdot}7 \times R_2^2)$$

or $$2{,}130 = 2{,}060 + 0{\cdot}7 R_2^2$$

or $$R_2^2 = 100$$

$$R_2 = 10 \text{ cm.}$$

$\therefore$ Diameter of hole $= 20$ cm.

13. The cast-iron cylindrical case of a friction clutch is 48 cm internal diameter and 2 cm thick. The internal radial pressure of the friction blocks on the case is 6 kg/cm², and the case makes about its axis 500 revolutions per minute. Estimate the greatest intensity of tensile stress in the material of the case, which may be taken as a thin shell. Weight of cast iron 7·2 g/cm³.

Due to internal pressure :

$$f = \frac{pD}{2t} = \frac{6 \times 48}{2 \times 2} = 72 \text{ kg/cm}^2$$

Due to rotation :

Mean diameter $= 48 + 2 = 50$ cm

$$f = \frac{\rho v^2}{g} = \frac{7{\cdot}2 \times \left(\dfrac{\pi \times 50 \times 500}{60}\right)^2}{1{,}000 \times 981}$$

$$= 12{\cdot}6 \text{ kg/cm}^2$$

Total tensile stress $= 72 + 12{\cdot}6$

$$= 84{\cdot}6 \text{ kg/cm}^2.$$

14. A built-up ring consists of an inner copper ring and an outer steel ring. The diameter of the surface of contact of the two rings is 60 cm. Determine the stresses set up in the steel and the copper by rotation of the ring at 3,000 r. p. m. Both the rings are of rectangular cross-section 12 mm in the radial direction and 20 mm in the direction perpendicular to the plane of ring.

For steel, $E = 2 \times 10^6$ kg/cm², $\rho = 7{\cdot}8$ g/cm³.

For copper, $E = 10^6$ kg/cm², $\rho = 8{\cdot}9$ g/cm³.

Due to the fact that copper has a greater density and a smaller modulus of elasticity than steel, the copper ring will press on the steel ring during rotation. Let p denote the pressure per sq. cm of contact surface between the two rings.

Due to contact pressure :

Stress in steel, $f_s = \dfrac{pD}{2t} = \dfrac{p \times 60}{2 \times 1{\cdot}2} = 25p$ tensile

Stress in copper, $f_c = \dfrac{pD}{2t} = \dfrac{p \times 60}{2 \times 1{\cdot}2} = 25p$ comp.

Due to centrifugal force :

Stress in steel

$$f_s' = \frac{\rho_s v^2}{g} = \frac{7{\cdot}8 \times \left(\dfrac{\pi \times 61{\cdot}2 \times 3{,}000}{60}\right)^2}{1{,}000 \times 981}$$

$= 735$ kg/cm² tensile

Stress in copper

$$f_c' = \frac{\rho_c v^2}{g} = \frac{8{\cdot}9 \times \left(\dfrac{\pi \times 58{\cdot}8 \times 3{,}000}{60}\right)^2}{1{,}000 \times 981}$$

$= 774$ kg/cm² tensile

Total stress in steel $= 735 + 25p$

Total stress in copper $= 774 - 25p$

Strain for the two rings will be equal.

Hence $$\frac{735 + 25p}{E_s} = \frac{774 - 25p}{E_c}$$

or $$\frac{735 + 25p}{2 \times 10^6} = \frac{774 - 25p}{10^6}$$

or $$p = 10{\cdot}84 \text{ kg/cm}^2$$

Stress in steel $= 735 + 25p =$ **1,006 kg/cm² tensile**

Stress in copper $= 774 - 25p =$ **503 kg/cm² tensile.**

15. Referring to Problem 14, determine the number of revolutions per minute at which the stress in the copper ring becomes equal to zero if the initial stress in the copper ring due to shrinkage is 400 kg/cm² compression.

$f_s = 25p$ tensile

$f_c = 25p$ comp.

$$f_s' = \frac{7{\cdot}8 \times \left(\dfrac{\pi \times 61{\cdot}2N}{60}\right)^2}{1{,}000 \times 981} = 81{\cdot}7 \times 10^{-6} N^2 \text{ kg/cm}^2 \text{ tensile}$$

$$f_c' = \frac{8{\cdot}9 \times \left(\dfrac{\pi \times 58{\cdot}8N}{60}\right)^2}{1{,}000 \times 981} = 86 \times 10^{-6} N^2 \text{ kg/cm}^2 \text{ tensile}$$

Hence

$$\frac{81 \cdot 7 \times 10^{-6} N^2 + 25p}{2 \times 10^6} = \frac{86 \times 10^{-6} N^2 - 25p}{10^6}$$

$$\therefore \quad p = 1 \cdot 204 \times 10^{-6} N^2$$

Stress in copper due to rotation

$$= 86 \times 10^{-6} N^2 - 25p = 55 \cdot 9 \times 10^{-6} N^2 \text{ kg/cm}^2 \text{ tensile}$$

Initial stress in copper due to shrinkage

$$= 400 \text{ kg/cm}^2 \text{ comp.}$$

For zero stress in copper

$$55 \cdot 9 \times 10^{-6} N^2 = 400$$

$$\therefore \quad N = \mathbf{2{,}680 \text{ r. p. m.}}$$

16. A steel disc of 20 cm outside diameter and 4 cm inside diameter is shrunk on a steel shaft so that the pressure between the shaft and disc at stand still is 500 kg/cm².

Assuming that the change in dimensions of the shaft are negligible, find the speed at which the disc loosens from the shaft.

$$\rho = 7 \cdot 8 \text{ g/cm}^3, \quad \frac{1}{m} = 0 \cdot 3.$$

At standstill the hoop and radial stresses in a thin disc are given by the equations

$$f = A + \frac{B}{r^2} \text{ tensile}$$

$$p = A - \frac{B}{r^2} \text{ tensile}$$

They are identical with the equations for a thick cylinder.

Due to shrinkage :

Hoop stress at the inside

$$f = \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2} \cdot p = \frac{10^2 + 2^2}{10^2 - 2^2} \times 500$$

$$= 542 \text{ kg/cm}^2 \text{ tensile}$$

$$\text{Hoop strain} = \frac{1}{E}\left(f + \frac{p}{m}\right) = \frac{1}{E}(542 + 0\ 3 \times 500)$$

$$= \frac{692}{E}$$

Due to rotation :

There is no prsssure from inside.

Hoop stress at the inside

$$f=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$=\frac{7\cdot8\omega^2}{1{,}000\times4\times981}(3\cdot3\times10^2+0\cdot7\times2^2)$$

$$=\frac{7\cdot8\times83\cdot2\omega^2}{1{,}000\times981}$$

$$\text{Hoop strain}=\frac{7\cdot8\times83\cdot2\omega^2}{1{,}000\times981\times E}$$

This hoop strain should be equal to the hoop strain due to shrinkage.

Hence $$\frac{7\cdot8\times83\cdot2\omega^2}{1{,}000\times981\times E}=\frac{692}{E}$$

or $$\omega^2=\frac{692\times1{,}000\times981}{7\cdot8\times83\cdot2}$$

$$\omega=1{,}023$$

$$N=\frac{60\omega}{2\pi}=\frac{60\times1{,}023}{2\pi}=\mathbf{9{,}770\ r.p.m.}$$

17. A circular disc of outside and inside radii r_1 and r_2 is made up in two parts, the common radius being a. The outer portion is shrunk on so as to exert a pressure on the inner. Prove that the hoop tension at the inside and outside of the disc will be equal for an angular velocity ω, if the shrinkage pressure at the common surface, when the disc is stationary, has the value

$$\frac{\rho\omega^2}{g}\quad\frac{(m+1)(r_1^2-a^2)(a^2-r_2^2)}{4ma^2}$$

Let p be the shrinkage pressure.

(a) **Due to shrinkage :**

Inner disc :

$$-p=A_1-\frac{B_1}{a^2}$$

$$0=A_1-\frac{B_1}{r_2^2}$$

Solving $$A_1=-\frac{a^2}{a^2-r_2^2}\cdot p,\ B_1=-\frac{r_2^2a^2}{a^2-r_2^2}\cdot p$$

At $\quad r=r_2,\ f=A_1+\dfrac{B_1}{r_2^2}=-\dfrac{2a^2p}{a^2-r_2^2}$(comp.) .. (1)

Outer disc :

$$-p=A_2-\frac{B_2}{a^2}$$

$$0=A_2-\frac{B_2}{r_1^2}$$

Solving $\quad A_2=\dfrac{a^2}{r_1^2-a^2}.p,\quad B_2=\dfrac{r_1^2a^2}{r_1^2-a^2}.p$

At $\quad r=r_1,\ f=A_2+\dfrac{B_2}{r_1^2}=\dfrac{2a^2p}{r_1^2-a^2}$ (tension) .. (2)

Due to rotation :

Consider the tube as a whole.

$$0=A_3-\frac{B_3}{r_1^2}-\left(3+\frac{1}{m}\right)\frac{\rho\omega^2}{8g}r_1^2$$

$$0=A_3-\frac{B_3}{r_2^2}-\left(3+\frac{1}{m}\right)\frac{\rho\omega^2}{8g}r_2^2$$

Solving $\quad A_3=\left(3+\dfrac{1}{m}\right)\dfrac{\rho\omega^2}{8g}(r_1^2+r_2^2)$

$$B_3=\left(3+\frac{1}{m}\right)\frac{\rho\omega^2}{8g}r_1^2r_2^2$$

At $\quad r=r_2,\ f=A_3+\dfrac{B_3}{r_2^2}-\left(1+\dfrac{3}{m}\right)\dfrac{\rho\omega^2}{8g}r_2^2$

$$=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}\text{ tension .. (3)}$$

At $\quad r=r_1,\quad f=A_3+\dfrac{B_3}{r_1^2}-\left(1+\dfrac{3}{m}\right)\dfrac{\rho\omega^2}{8g}r_1^2$

$$=\frac{\rho\omega^2}{4g}\left\{\left(1-\frac{1}{m}\right)r_1^2+\left(3+\frac{1}{m}\right)r_2^2\right\}$$

tension .. (4)

Total hoop tension at the inside = (1) + (3)

Total hoop tension at the outside = (2) + (4)

Hence $$-\frac{2a^2p}{a^2-r_2^2}+\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}$$

$$=\frac{2a^2p}{r_1^2-a^2}+\frac{\rho\omega^2}{4g}\left\{\left(1-\frac{1}{m}\right)r_1^2+\left(3+\frac{1}{m}\right)r_2^2\right\}$$

or $$\frac{\rho\omega^2}{4g}\left\{\left(2+\frac{2}{m}\right)r_1^2-\left(2+\frac{2}{m}\right)r_2^2\right\}$$

$$=2a^2p\left\{\frac{1}{a^2-r_2^2}+\frac{1}{r_1^2-a^2}\right\}$$

or $$\left(1+\frac{1}{m}\right)\frac{\rho\omega^2}{2g}(r_1^2-r_2^2)=\frac{2a^2p(r_1^2-r_2^2)}{(r_1^2-a^2)(a^2-r_2^2)}$$

$$\therefore \quad p=\frac{\rho\omega^2}{g}\cdot\frac{(m+1)(r_1^2-a^2)(a^2-r_2^2)}{4ma^2}.$$

18. A thin circular disc of external radius r_1 is forced on to a rigid shaft of radius r_2. Prove that when the speed is ω the pressure between the disc and the shaft will be reduced by

$$\frac{\rho\omega^2}{g}\cdot\frac{(r_1^2-r_2^2)\{(3m+1)r_1^2+(m-1)r_2^2\}}{4\{(m+1)r_1^2+(m-1)r_2^2\}}.$$

Due to shrinkage :

Let p be the pressure at the common surface.

At the inside, $f=\frac{r_1^2+r_2^2}{r_1^2-r_2^2}.p$ tension

$$\text{Hoop strain}=\frac{1}{E}\left(f+\frac{p}{m}\right)=\frac{p}{E}\left(\frac{r_1^2+r_2^2}{r_1^2-r_2^2}+\frac{1}{m}\right)$$

Due to rotation :

Let p be the pressure at the common surface.

At the inside stresses are—

Due to $p'=\frac{r_1^2+r_2^2}{r_1^2-r_2^2}.p'$ tension

Due to centrifugal force

$$=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\} \text{ tension}$$

Total tensile stress

$$f=\frac{r_1^2+r_2^2}{r_1^2-r_2^2}.p'+\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}$$

$$\text{Hoop strain} = \frac{1}{E}\left(f + \frac{p'}{m}\right)$$

$$= \frac{1}{E}\left[\left(\frac{r_1^2+r_2^2}{r_1^2-r_2^2}+\frac{1}{m}\right)p' + \frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}\right]$$

Hoop strains will be equal.

$$\therefore \quad \frac{p}{E}\left(\frac{r_1^2+r_2^2}{r_1^2-r_2^2}+\frac{1}{m}\right)$$

$$= \frac{1}{E}\left[\left(\frac{r_1^2+r_2^2}{r_1^2-r_2^2}+\frac{1}{m}\right)p' + \frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}\right]$$

$$\text{or} \quad (p-p')\left(\frac{r_1^2+r_2^2}{r_1^2-r_2^2}+\frac{1}{m}\right) = \frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)r_1^2+\left(1-\frac{1}{m}\right)r_2^2\right\}$$

$$\text{or} \quad (p-p')\frac{(m+1)r_1^2+(m-1)r_2^2}{m(r_1^2-r_2^2)} = \frac{\rho\omega^2}{4g}\cdot\frac{(3m+1)r_1^2+(m-1)r_2^2}{m}$$

$$\therefore \quad p-p' = \frac{\rho\omega^2}{g}\cdot\frac{(r_1^2-r_2^2)\{(3m+1)r_1^2+(m-1)r_2^2\}}{4\{(m+1)r_1^2+(m-1)r_2^2\}}.$$

19. Determine the stresses due to centrifugal force in a steel rotor with an outer radius of 60 cm and with the radius of the inner hole equal to 10 cm. The outer portion of the rotor is cut by slots 20 cm deep for the windings (Fig. 26). The rotor revolves at 1,800 r.p.m. The weight of the windings in the slots is the same as that of the material removed.

$\rho = 7{\cdot}8$ **g/cm³**, $m = \frac{10}{3}$.

Because of the radial slots, the part of the rotor between the outer radius and the 40 cm radius can support no tensile hoop stresses. The centrifugal force due to this rotating ring is transmitted as a radial tensile stress across the surface of the disc of 40 cm radius.

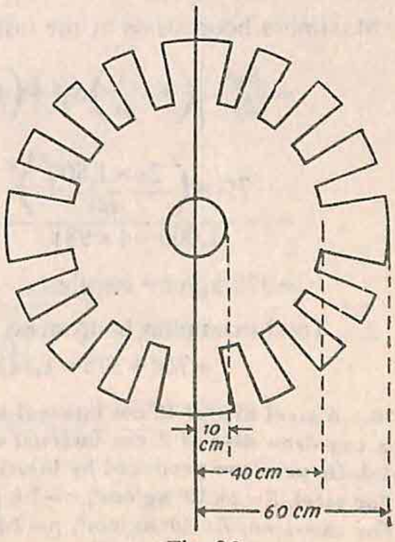

Fig. 26

Centrifugal force

$$= \int_{40}^{60} \frac{2\pi r.dr.t.\rho.r\omega^2}{g}$$

$$= \frac{\rho\omega^2}{g}.2\pi t\int_{40}^{60} r^2 dr$$

$$=\frac{\rho\omega^2}{g}\cdot\frac{2\pi t}{3}(60^3-40^3)=\frac{\rho\omega^2}{g}\cdot\frac{2\pi t\times 152{,}000}{3}$$

Area on which this acts $=2\pi\times 40\times t$

$\therefore$ Radial tensile stress

$$p=\frac{\rho\omega^2}{g}\times\frac{2\pi t\times 152{,}000}{3\times 2\pi\times 40\times t}=\frac{\rho\omega^2}{g}\times\frac{3{,}800}{3}$$

$$=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times 1{,}800}{60}\right)^2\times 3{,}800}{1{,}000\times 981\times 3}$$

$$=358\ \text{kg/cm}^2$$

Now we can treat the portion of the rotor between radii of 10 cm and 40 cm as a rotating disc with a radial tensile stress of 358 kg/cm² on the external surface.

Due to external pressure :

Maximum hoop stress at the inside

$$=\frac{2R_1^2p}{R_1^2-R_2^2}=\frac{2\times 40^2\times 358}{40^2-10^2}$$

$$=764\ \text{kg/cm}^2\ \text{tensile.}$$

Due to centrifugal force :

Maximum hoop stress at the inside

$$=\frac{\rho\omega^2}{4g}\left\{\left(3+\frac{1}{m}\right)R_1^2+\left(1-\frac{1}{m}\right)R_2^2\right\}$$

$$=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times 1{,}800}{60}\right)^2}{1{,}000\times 4\times 981}\times(3{\cdot}3\times 40^2+0{\cdot}7\times 10^2)$$

$$=378\ \text{kg/cm}^2\ \text{tensile.}$$

$\therefore$ Total maximum hoop stress at the inside

$$=764+378=\mathbf{1{,}142\ kg/cm^2\ tensile.}$$

20. A steel disc of 10 cm internal and 20 cm external radius is shrunk on a cast-iron disc of 2 cm internal radius. Determine the change in the shrink-fit pressure produced by inertia forces at 3,600 r.p.m.

For steel, $E=2\times 10^6$ kg/cm², $\rho=7{\cdot}8$ g/cm³.

For cast-iron, $E=10^6$ kg/cm², $\rho=7{\cdot}2$ g/cm³.

Poisson's ratio 0·3 for both.

Let p_0 be the increase in pressure between the steel and cast-iron discs.

Outer steel disc :

$$\frac{\rho\omega^2}{g}=\frac{7{\cdot}8\times\left(\dfrac{2\pi\times 3{,}600}{60}\right)^2}{1{,}000\times 981}=1{\cdot}13$$

$$-p_0=A_1-\frac{B_1}{100}-(3+0{\cdot}3)\times\frac{1{\cdot}13\times 100}{8}$$

$$=A_1-\frac{B_1}{100}-46{\cdot}6 \qquad \ldots \qquad (1)$$

$$0=A_1-\frac{B_1}{400}-(3+0{\cdot}3)\times\frac{1{\cdot}13\times 400}{8}$$

$$=A_1-\frac{B_1}{400}-186{\cdot}4 \qquad \ldots \qquad (2)$$

Solving $A_1=\frac{1}{3}(p_0+699)$, $B_1=\frac{400}{3}(p_0+139{\cdot}8)$

At the inner surface

$$f=A_1+\frac{B_1}{100}-(1+3\times 0{\cdot}3)\times\frac{1{\cdot}13\times 100}{8}$$

$$=\tfrac{1}{3}(p_0+699)+\tfrac{4}{3}(p_0+139{\cdot}8)-26{\cdot}8$$

$$=\tfrac{5}{3}p_0+393$$

$$\text{Hoop strain}=\frac{1}{E}\left(f+\frac{p_0}{m}\right)=\frac{1}{2\times 10^6}\left(\frac{5}{3}p_0+393+0{\cdot}3p_0\right)$$

$$=\frac{1}{2\times 10^6}\left(\frac{5{\cdot}9}{3}p_0+393\right)$$

Inner cast-iron disc :

$$\frac{\rho\omega^2}{g}=\frac{7{\cdot}2\times\left(\dfrac{2\pi\times 3{,}600}{60}\right)^2}{1{,}000\times 981}=1{\cdot}043$$

$$-p_0=A_2-\frac{B_2}{100}-(3+0{\cdot}3)\times\frac{1{\cdot}043\times 100}{8}$$

$$=A_2-\frac{B_2}{100}-43 \qquad \ldots \qquad (3)$$

$$0 = A_2 - \frac{B_2}{4} - (3+0\cdot3) \times \frac{1\cdot043 \times 4}{8}$$

$$= A_2 - \frac{B_2}{4} - 1\cdot72 \quad \ldots \quad (4)$$

Solving $A_2 = \frac{25}{24}(43 - p_0)$, $B_2 = \frac{25}{6}(41\cdot3 - p_0)$

At the outer surface

$$f = A_2 + \frac{B_2}{100} - (1 + 3 \times 0\cdot3) \times \frac{1\cdot043 \times 100}{8}$$

$$= \frac{25}{24}(43 - p_0) + \frac{1}{24}(41\cdot3 - p_0) - 24\cdot8$$

$$= 21\cdot7 - \frac{13}{12} p_0$$

$$\text{Hoop strain} = \frac{1}{E}\left(f + \frac{p_0}{m}\right) = \frac{1}{10^6}\left(21\cdot7 - \frac{13}{12}p_0 + 0\cdot3p_0\right)$$

$$= \frac{1}{10^6}\left(21\cdot7 - \frac{4\cdot7}{6}p_0\right)$$

Hoop strain of the two discs at the common surface will be equal.

Hence $$\frac{1}{2 \times 10^6}\left(\frac{5\cdot9}{3}p_0 + 393\right) = \frac{1}{10^6}\left(21\cdot7 - \frac{4\cdot7}{6}p_0\right)$$

or, $$\frac{10\cdot6}{3}p_0 = -349\cdot6$$

$$\therefore \quad p_0 = -98\cdot9 \text{ kg/cm}^2 \text{ (decrease).}$$

BENDING OF CURVED BARS

1. A curved bar of rectangular section, initially unstressed, is subjected to a bending moment of 150 kg m which tends to straighten the bar.

The section is 4 cm wide by 5 cm deep in the plane of bending, and the mean radius of curvature is 10 cm. Find the position of the neutral axis and magnitudes of the greatest bending stresses and draw a diagram to show approximately how the stress varies across the section. (Lond. Univ.)

$$\text{Modified area, } A' = Rb \log_e \frac{2R+d}{2R-d} = 10 \times 4 \times \log_e \frac{25}{15}$$

$$= 40 \times 0{\cdot}5108 = 20{\cdot}432 \text{ cm}^2$$

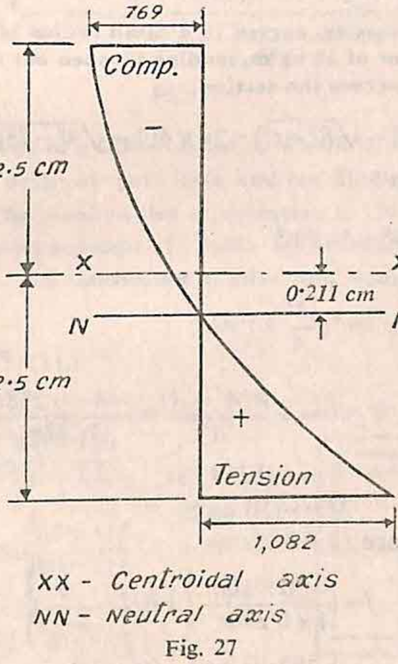

Fig. 27

$$A = 4 \times 5 = 20 \text{ cm}^2$$

$$\therefore \quad A' - A = 0{\cdot}432 \text{ cm}^2, \frac{A'}{A} = 1{\cdot}0216$$

$M = -150$ kg m (tending to decrease the curvature)

$$h=-\frac{R(A'-A)}{A'}=-\frac{10\times 0{\cdot}432}{20{\cdot}432}=\mathbf{-0{\cdot}211\ cm}$$

$$f=\frac{M}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+y}\right\}$$

At the inside face ($y=-2{\cdot}5$ cm)

$$f=\frac{-150\times 100}{10\times 0{\cdot}432}\left\{1{\cdot}0216-\frac{10}{10-2{\cdot}5}\right\}$$

$$=\mathbf{1{,}082\ kg/cm^2\ (tensile)}$$

At the outside face ($y=+2{\cdot}5$ cm)

$$f=\frac{-150\times 100}{10\times 0{\cdot}432}\left\{1{\cdot}0216-\frac{10}{10+2{\cdot}5}\right\}$$

$$=\mathbf{-769\ kg/cm^2\ (compressive)}$$

The stress distribution is shown in Fig. 27.

2. A bar 4 cm diameter, curved to a mean radius of 4 cm, is subjected to a bending moment of 50 kg m, tending to open out the bend. Plot the stress distribution across the section. (Lond. Univ.)

$$A'=2\pi R(R-\sqrt{R^2-r^2})=2\pi\times 4(4-\sqrt{4^2-2^2})$$
$$=4{\cdot}288\pi\ \text{cm}^2$$

$$A=\frac{\pi}{4}\times 4^2=4\pi\ \text{cm}^2$$

$$A'-A=0{\cdot}288\pi\ \text{cm}^2;\ \frac{A'}{A}=1{\cdot}072$$

$$h=-\frac{R(A'-A)}{A'}=\frac{-4\times 0{\cdot}288\pi}{4{\cdot}288\pi}$$
$$=-0{\cdot}269\ \text{cm}$$
$$M=-50\ \text{kg m}$$

At the inside face ($y=-2$ cm)

$$f=\frac{-50\times 100}{4\times 0{\cdot}288\pi}\left\{1{\cdot}072-\frac{4}{4-2}\right\}$$

$$=1{,}282\ \text{kg/cm}^2\quad \text{(tensile)}$$

At the outside face ($y=+2$ cm)

$$f=\frac{-50\times 100}{4\times 0{\cdot}288\pi}\left\{1{\cdot}072-\frac{4}{4+2}\right\}$$

$$=-560\ \text{kg/cm}^2\ \text{(compressive)}$$

The stress distribution is shown in Fig. 28.

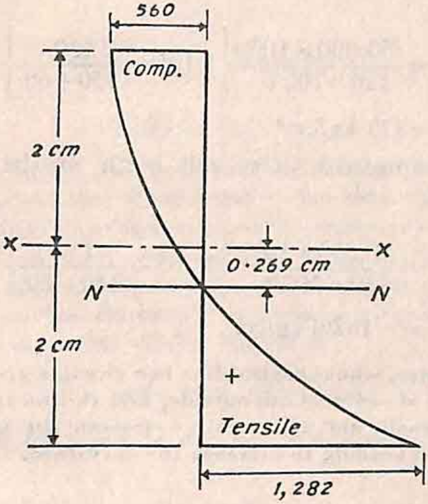

Fig. 28

3. The cross-section of a curved beam is a hollow rectangle of outside dimensions 120 cm deep × 80 cm wide and inside dimensions 115 cm deep × 75 cm wide. The mean radius of curvature is 150 cm and the beam is subjected to a bending moment of 50,000 kg m tending to increase the curvature. Calculate the maximum tensile and compressive stresses set up in the beam.

$A = 80 \times 120 - 75 \times 115$
$= 975 \text{ cm}^2$

$A' = 150 \times 80 \log_e \dfrac{300+120}{300-120}$

$- 150 \times 75 \log_e \dfrac{300+115}{300-115}$

$= 150(80 \times 0{\cdot}8473$
$- 75 \times 0{\cdot}8080)$

$= 1077{\cdot}6 \text{ cm}^2$

$A' - A = 102{\cdot}6 \text{ cm}^2$

$\dfrac{A'}{A} = 1{\cdot}105$

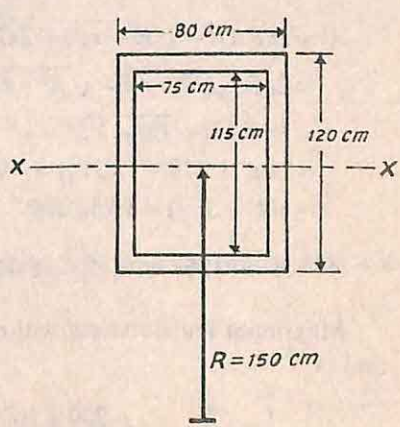

Fig. 29

$M = +50{,}000$ kg m (tending to increase the curvature)

Maximum tensile stress will occur at the outer surface, $y=+60$ cm and is

$$=\frac{50{,}000\times100}{150\times102{\cdot}6}\left\{1{\cdot}105-\frac{150}{150+60}\right\}$$

$$=\mathbf{127\ kg/cm^2}$$

Maximum compressive stress will occur at the inside surface, $y=-60$ cm and is

$$=\frac{50{,}000\times100}{150\times102{\cdot}6}\left\{1{\cdot}105-\frac{150}{150-60}\right\}$$

$$=\mathbf{-182{\cdot}6\ kg/cm^2}.$$

4. A curved beam, whose central line is a circular arc of radius 12 cm, is formed of a tube of radius 4 cm outside, and thickness 5 mm. Determine the greatest tensile and compressive stresses set up by a bending moment of 200 kg m tending to increase the curvature.

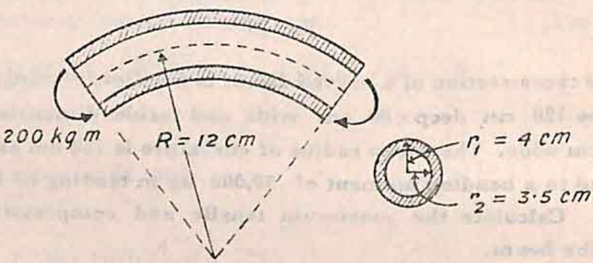

Fig. 30

$$A'=2\pi R\,(R-\sqrt{R^2-r_1^{\,2}})-2\pi R(R-\sqrt{R^2-r_2^{\,2}})$$

$$=2\pi R(\sqrt{R^2-r_2^{\,2}}-\sqrt{R^2-r_1^{\,2}})$$

$$=2\pi\times12(\sqrt{12^2-3{\cdot}5^2}-\sqrt{12^2-4^2})$$

$$=24\pi(11{\cdot}478-11{\cdot}314)=3{\cdot}936\pi\ \text{cm}^2$$

$$A=\pi(4^2-3{\cdot}5^2)=3{\cdot}75\pi\ \text{cm}^2$$

$$A'-A=0{\cdot}186\pi\ \text{cm}^2,\ \frac{A'}{A}=1{\cdot}0496$$

Maximum tensile stress will occur at the outer edge, $y=+4$ cm and is

$$=\frac{200\times100}{12\times0{\cdot}186\pi}\left\{1{\cdot}0496-\frac{12}{12+4}\right\}$$

$$=\mathbf{855\ kg/cm^2}$$

Maximum compressive stress will occur at the inner edge, $y=-4$ cm and is

$$=\frac{200\times 100}{12\times 0{\cdot}186\pi}\left\{1{\cdot}0496-\frac{12}{12-4}\right\}$$

$$=-1{,}285\ \text{kg/cm}^2.$$

5. A bar of rectangular cross-section, breadth b and depth d, has initial radius of curvature R measured to the axis through the centroid of the section. The plane of curvature is parallel to two of the faces of the rectangle. Pure couples of magnitude M are applied to the ends of the bar tending to increase the curvature. Assuming that plane sections remain plane, show that the neutral axis will be a distance h from the axis through the centroid of the section such that

$$\frac{d}{R-h}=\log_e\frac{2R+d}{2R-d}.$$

Show also that the stress in the bar at a distance y from the neutral axis is

$$\frac{M}{bdh}\cdot\frac{y}{R-h+y}$$

$$h\ (\text{numerical})=\frac{R(A'-A)}{A'}=R-\frac{RA}{A'}$$

or $$R-h=\frac{RA}{A'}$$

or $$\frac{1}{R-h}=\frac{A'}{RA}=\frac{Rb\log_e\dfrac{2R+d}{2R-d}}{Rbd}$$

or $$\frac{d}{R-h}=\log_e\frac{2R+d}{2R-d}$$

$$f=\frac{M}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+y_x}\right\}$$

In the above equation y_x is measured from the centroidal axis XX.

But $$A'=\frac{RA}{R-h}$$

$$\therefore\ f=\frac{M}{R\left(\dfrac{RA}{R-h}-A\right)}\left\{\frac{R}{R-h}-\frac{R}{R+y_x}\right\}$$

$$=\frac{M(R-h)}{RAh}\cdot\frac{R(h+y_x)}{(R-h)(R+y_x)}$$

$$=\frac{M}{bdh}\cdot\frac{h+y_x}{R+y_x}$$

If y is measured from the neutral axis NN

$$y=h+y_x$$

Hence $$f=\frac{M}{bdh}\cdot\frac{y}{R-h+y}.$$

6. A frame of rectangular section is subjected to a load of 2,000 kg as shown in Fig. 31.

(a) Use the ordinary beam theory for straight beams to determine to the nearest millimetre the dimension d at the section XX for a maximum tensile stress of 900 kg/cm².

(b) Calculate now the normal stresses at the inner and outer edges at XX for the section found, but this time allow for the curvature of the beam.

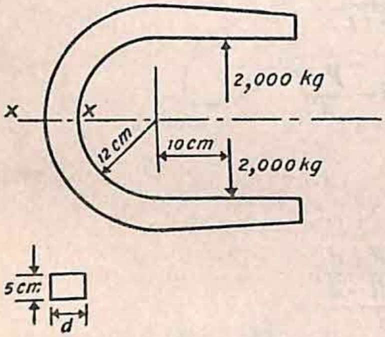

Fig. 31

(a) Treated as a simple problem in eccentric loading in one plane—

Eccentricity $e=10+12+\frac{d}{2}$

$$=22+\frac{d}{2}\text{ cm}$$

Direct stress

$$=\frac{P}{A}=\frac{2{,}000}{5d}=\frac{400}{d}\text{ tensile}$$

$$\text{Bending stress}=\frac{P.e}{I}\cdot\frac{d}{2}=\frac{2{,}000\left(22+\frac{d}{2}\right)}{\frac{5d^3}{12}}\cdot\frac{d}{2}$$

$$=\frac{52{,}800}{d^2}+\frac{1{,}200}{d}$$

Hence maximum tensile stress

$$=\frac{400}{d}+\frac{52{,}800}{d^2}+\frac{1{,}200}{d}$$

$$=\frac{1{,}600}{d}+\frac{52{,}800}{d^2}$$

$$\therefore \quad \frac{1{,}600}{d}+\frac{52{,}800}{d^2}=900$$

or $$9d^2-16d-528=0$$

Solving $$d=\mathbf{8{\cdot}6\ cm.}$$

(b) Allowing for the curvature of the beam—

$$R=12+4{\cdot}3=16{\cdot}3\text{ cm}$$

$$A'=16{\cdot}3\times5\times\log_e\frac{32{\cdot}6+8{\cdot}6}{32{\cdot}6-8{\cdot}6}=81{\cdot}5\log_e\frac{10{\cdot}3}{6}$$

$$=81{\cdot}5\times0{\cdot}5404$$

$$=44{\cdot}04\text{ cm}^2$$

$$A=5\times8{\cdot}6=43\text{ cm}^2$$

$$A'-A=1{\cdot}04\text{ cm}^2,\quad \frac{A'}{A}=1{\cdot}024$$

$$M=-2{,}000\times(10+12+4{\cdot}3)=-52{,}600\text{ kg cm}$$

At the inner edge, $y=-4{\cdot}3$ cm

$$f=\frac{-52{,}600}{16{\cdot}3\times1{\cdot}04}\left\{1{\cdot}024-\frac{16{\cdot}3}{16{\cdot}3-4{\cdot}3}\right\}+\frac{2{,}000}{43}$$

$$=1{,}036+47=\mathbf{1{,}083\ kg/cm^2\ tensile.}$$

At the outer edge, $y=+4{\cdot}3$ cm

$$f=\frac{-52{,}600}{16{\cdot}3\times1{\cdot}04}\left\{1{\cdot}024-\frac{16{\cdot}3}{16{\cdot}3+4{\cdot}3}\right)+\frac{2{,}000}{43}$$

$$=-723+47=\mathbf{-676\ kg/cm^2\ comp.}$$

7. A crane hook whose horizontal cross-section through the centre of curvature is trapezoidal, 4 cm wide at the inside and 1 cm wide at the outside, thickness 4 cm, carries a vertical load of 1,000 kg whose line of action passes through the centre of curvature and is 3 cm from the inside edge of the section. Calculate the maximum tensile and compressive stresses in the section.

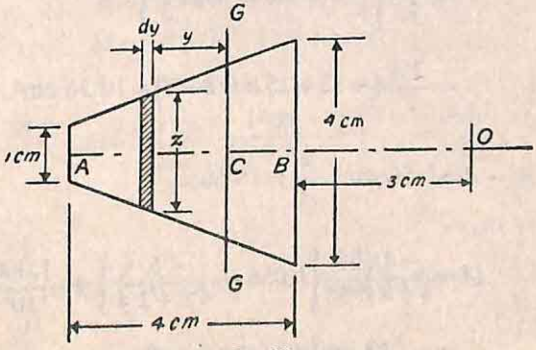

Fig. 32

$$A = \tfrac{1}{2}(1+4)\times 4 = 10\text{ cm}^2$$

Taking moment about the inner face

$$10 \times BC = 4 \times 2 + \tfrac{1}{2} \times \tfrac{3}{2} \times 4 \times \tfrac{4}{3} \times 2$$

$$\therefore \quad BC = 1{\cdot}6 \text{ cm}$$

$$AC = 2{\cdot}4 \text{ cm}$$

$$R = OC = 4{\cdot}6 \text{ cm}$$

$$M = -1{,}000 \times 4{\cdot}6 = -4{,}600 \text{ kg cm}$$

$$A' = \int \frac{R\delta A}{R+y} = 4{\cdot}6 \int \frac{z\,dy}{4{\cdot}6+y}$$

$$z = 1 + \frac{3}{4}(2{\cdot}4 - y) = \frac{11{\cdot}2 - 3y}{4}$$

$$A' = 4{\cdot}6 \int_{-1\cdot6}^{+2\cdot4} \frac{\frac{11\cdot2-3y}{4}\,dy}{4{\cdot}6+y} = \frac{2{\cdot}3}{2}\int_{-1\cdot6}^{+2\cdot4} \frac{11{\cdot}2-3y}{4{\cdot}6+y}\,dy$$

$$= \frac{2{\cdot}3}{2}\int_{-1\cdot6}^{+2\cdot4}\left[-3 + \frac{25}{4{\cdot}6+y}\right]dy$$

$$= \frac{2{\cdot}3}{2}\Big[-3y + 25\log_e(4{\cdot}6+y)\Big]_{-1\cdot6}^{+2\cdot4}$$

$$= \frac{2{\cdot}3}{2}\left[-3\times 4 + 25\log_e\frac{7}{3}\right]$$

$$= \frac{2{\cdot}3}{2}[-12 + 25 \times 0{\cdot}8473] = 10{\cdot}56 \text{ cm}^2$$

$$A' - A = 0{\cdot}56 \text{ cm}^2, \frac{A'}{A} = 1{\cdot}056$$

Stress at $\quad A = \dfrac{-4{,}600}{4{\cdot}6 \times 0{\cdot}56}\left\{1{\cdot}056 - \dfrac{4{\cdot}6}{4{\cdot}6+2{\cdot}4}\right\} + \dfrac{1{,}000}{10}$

$$= -613 \text{ kg/cm}^2 \textbf{ (compressive)}$$

Stress at $\quad B = \dfrac{-4{,}600}{4{\cdot}6 \times 0{\cdot}56}\left\{1{\cdot}056 - \dfrac{4{\cdot}6}{4{\cdot}6 - 1{\cdot}6}\right\} + \dfrac{1{,}000}{10}$

$= +952$ kg/cm² (tensile).

8. The principal section of a hook is a symmetrical trapezium 4·5 cm deep, the width at the inside of the hook being 4 cm and at the outside 2 cm. The centre of curvature of both inside and outside of the hook at this section is in the plane of the section and 4 cm from the inside of it, and the load line passes 3·5 cm from the inner side of the section. Estimate the safe load for this hook in order that the greatest tensile stress shall not exceed 1,200 kg/cm².

(Ans. 1,642 kg.)

9. A hook with a *T* section has a flange 10 cm wide and 3 cm deep, and a web 3 cm wide and 15 cm deep, with the flange on the inside. The radius of curvature of the inner surface is 10 cm. The hook carries a load of 6,000 kg whose line of action is 12 cm from the inside edge of the section. Estimate the maximum tensile and compressive stresses in the section.

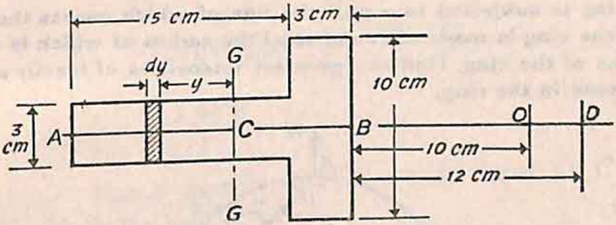

Fig. 33

$$A = 10 \times 3 + 15 \times 3 = 75 \text{ cm}^2$$

Taking moment about the inner face

$$75 \times BC = 10 \times 3 \times \tfrac{3}{2} + 15 \times 3 \times \tfrac{21}{2}$$

$$\therefore \quad BC = 6{\cdot}9 \text{ cm}$$

$$AC = 11{\cdot}1 \text{ cm}$$

$$R = 16{\cdot}9 \text{ cm}, \; l = CD = 18{\cdot}9 \text{ cm}$$

$$M = -6{,}000 \times 18{\cdot}9 \text{ kg cm}$$

$$A' = \int \frac{R\delta A}{R+y} = 16{\cdot}9 \left\{ \int_{-6\cdot9}^{-3\cdot9} \frac{10dy}{16{\cdot}9 + y} + \int_{-3\cdot9}^{+11\cdot1} \frac{3dy}{16{\cdot}9 + y} \right\}$$

$$= 16{\cdot}9 \left\{ 10 \log_e \Big(16{\cdot}9 + y\Big)_{-6\cdot9}^{-3\cdot9} + 3 \log_e \Big(16{\cdot}9 + y\Big)_{-3\cdot9}^{+11\cdot1} \right\}$$

$$=16{\cdot}9\left(10\log_e\tfrac{13}{10}+3\log_e\tfrac{28}{13}\right)$$
$$=16{\cdot}9(10\times0{\cdot}2624+3\times0{\cdot}7672)$$
$$=83{\cdot}24\text{ cm}^2$$

$$A'-A=8{\cdot}24\text{ cm}^2,\ \frac{A'}{A}=1{\cdot}11$$

Maximum tensile stress at B

$$=\frac{-6{,}000\times18{\cdot}9}{16{\cdot}9\times8{\cdot}24}\left\{1{\cdot}11-\frac{16{\cdot}9}{16{\cdot}9-6{\cdot}9}\right\}+\frac{6{,}000}{75}$$
$$=+552\text{ kg/cm}^2$$

Maximum compressive stress at A

$$=\frac{-6{,}000\times18{\cdot}9}{16{\cdot}9\times8{\cdot}24}\left\{1{\cdot}11-\frac{16{\cdot}9}{16{\cdot}9+11{\cdot}1}\right\}+\frac{6{,}000}{75}$$
$$=-332\text{ kg/cm}^2.$$

10. A ring is subjected to a pull the line of which passes through its centre. If the ring is made of round steel the radius of which is $\frac{1}{3}$ of the mean radius of the ring, find the greatest intensities of tensile and compressive stress in the ring.

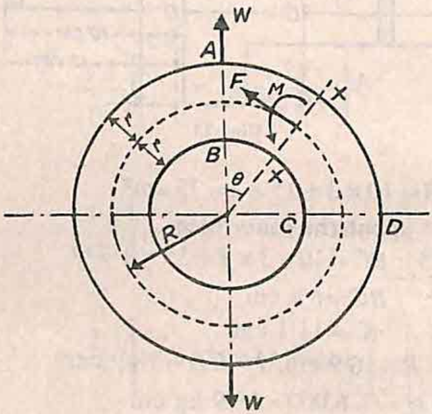

Fig. 34

$$R=3r$$
$$A'=2\pi R(R-\sqrt{R^2-r^2})=2\pi\times3r(3r-\sqrt{9r^2-r^2})$$
$$=1{\cdot}0296\pi r^2$$
$$A=\pi r^2$$

$$\therefore\quad A'-A=0{\cdot}0296\pi r^2,\ \frac{A'}{A}=1{\cdot}0296$$

At a section XX inclined at θ with the load line

$$F=\frac{W}{2}\sin\theta \text{ and } M=WR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$$

At $\theta=0$, $F=0$ and $M_0=\dfrac{WR}{\pi}$

At $\theta=\dfrac{\pi}{2}$, $F=\dfrac{W}{2}$ and $M_1=-WR\left(\dfrac{1}{2}-\dfrac{1}{\pi}\right)$

Maximum compressive stress at B (intrados, along the load line)

$$=\frac{M_0}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}$$

$$=\frac{W}{\pi(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}$$

$$=\frac{W}{\pi\times 0{\cdot}0296\pi r^2}\left\{1{\cdot}0296-\frac{3r}{2r}\right\}$$

$$=-\frac{\mathbf{1{\cdot}61W}}{r^2}$$

$R<3{\cdot}66r$. Hence maximum tension will occur at C (intrados, perpendicular to the load line).

Maximum tensile stress at C

$$=\frac{M_1}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}+\frac{W}{2A}$$

$$=\frac{-W\left(\frac{1}{2}-\frac{1}{\pi}\right)}{A'-A}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}+\frac{W}{2A}$$

$$=\frac{-W\left(\frac{1}{2}-\frac{1}{\pi}\right)}{0{\cdot}0296\pi r^2}\left\{1{\cdot}0296-\frac{3r}{2r}\right\}+\frac{W}{2\pi r^2}$$

$$=0{\cdot}919\frac{W}{r^2}+0{\cdot}159\frac{W}{r^2}$$

$$=\mathbf{1{\cdot}078}\,\frac{\mathbf{W}}{r^2}.$$

11. A ring is made of round steel 3 cm diameter, and the mean diameter of the ring is 15 cm. Estimate the greatest intensities of tensile and com-

pressive stress resulting from a pull of 600 kg on the ring. Also find the increase in diameter in the line of pull and decrease in diameter perpendicular to the line of pull. $E=2\times10^6$ kg/cm².

$R=7\cdot5$ cm, $r=1\cdot5$ cm

$$A'=2\pi R(R-\sqrt{R^2-r^2})=2\pi\times7\cdot5(7\cdot5-\sqrt{7\cdot5^2-1\cdot5^2})$$

$$=15\pi(7\cdot5-7\cdot3485)=2\cdot2725\pi \text{ cm}^2$$

$$A=\pi\times1\cdot5^2=2\cdot25\pi \text{ cm}^2$$

$$A'-A=0\cdot0225\pi,\ \frac{A'}{A}=1\cdot01$$

$$M=WR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$$

Along the load line ($\theta=0$), $M_0=\dfrac{WR}{\pi}$

Perpendicular to the load line $\left(\theta=\dfrac{\pi}{2}\right)$

$$M_1=-WR\left(\frac{1}{2}-\frac{1}{\pi}\right)$$

Maximum compressive stress will occur at the intrados, along the load line and is

$$=\frac{M_0}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}=\frac{W}{\pi(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}$$

$$=\frac{600}{\pi\times0\cdot0225\pi}\left\{1\cdot01-\frac{7\cdot5}{6}\right\}$$

$$=-648 \text{ kg/cm}^2$$

$R>3\cdot66r$. Hence maximum tensile stress will occur at the extrados, along the load line and is

$$=\frac{M_0}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+r}\right\}=\frac{W}{\pi(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+r}\right\}$$

$$=\frac{600}{\pi\times0\cdot0225\pi}\left\{1\cdot01-\frac{7\cdot5}{9}\right\}$$

$$=478 \text{ kg/cm}^2$$

Increase in diameter along the load line

$$=0{\cdot}149\frac{WR^3}{EI}=\frac{0{\cdot}149\times600\times7{\cdot}5^3}{2\times10^6\times\frac{\pi}{64}\times81}$$

$$=0{\cdot}00474 \text{ cm}$$

Decrease in diameter perpendicular to the load line

$$=0{\cdot}137\frac{WR^3}{EI}=\frac{0{\cdot}137\times600\times7{\cdot}5^3}{2\times10^6\times\frac{\pi}{64}\times81}$$

$$=0{\cdot}00436 \text{ cm.}$$

12. A steel ring of 20 cm mean diameter has a rectangular cross-section 5 cm in the radial direction and 3 cm perpendicular to the radial direction. If the maximum tensile stress is limited to 1,200 kg/cm² and the deflection of the ring in the direction of loading is limited to 0·01 cm find the load that the ring can carry. $E=2\times10^6$ kg/cm².

$$R=10 \text{ cm},\ b=3 \text{ cm},\ d=5 \text{ cm}$$

$$A'=Rb\log_e\frac{2R+d}{2R-d}=10\times3\times\log_e\frac{25}{15}=30\times0{\cdot}5108$$

$$=15{\cdot}324 \text{ cm}^2$$

$$A=15 \text{ cm}^2$$

$$A'-A=0{\cdot}324 \text{ cm}^2,\ \frac{A'}{A}=1{\cdot}022$$

$$y=\frac{d}{2}=2{\cdot}5 \text{ cm}$$

$R>3{\cdot}66y$. Hence the maximum tensile stress will occur at the extrados, along the load line.

Along the load line ($\theta=0$), $M_0=\frac{WR}{\pi}$

Maximum tensile stress

$$=\frac{M_0}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+y}\right\}=\frac{W}{\pi(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+y}\right\}$$

$$1{,}200=\frac{W}{\pi\times0{\cdot}324}\left\{1{\cdot}022-\frac{10}{12{\cdot}5}\right\}$$

$\therefore\ W=5{,}500$ kg

Deflection of the ring along the load line

$$=0{\cdot}149\frac{WR^3}{EI}$$

Hence $$0{\cdot}01=\frac{0{\cdot}149\times W\times 1{,}000}{2\times 10^6\times \frac{1}{12}\times 3\times 125}$$

$$\therefore\quad W=4{,}190 \text{ kg}$$

Safe load = **4,190 kg.**

13. A ring of mean diameter 8 cm is made of round steel 2 cm diameter. It is subjected to four equal pulls in two directions at right angles and passing through the centre. Determine the maximum value of the pulls if the maximum tensile stress is not to exceed 800 kg/cm².

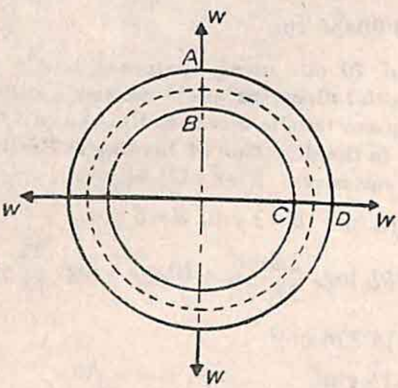

Fig. 35

$$A=\pi \text{ cm}^2$$

$$A'=2\pi\times 4(4-\sqrt{16-1})=8\pi(4-3{\cdot}873)$$
$$=1{\cdot}016\pi \text{ cm}^2$$

$$A'-A=0{\cdot}016\pi \text{ cm}^2,\quad \frac{A'}{A}=1{\cdot}016$$

Consider vertical load only.

At $\theta=0,\quad M_0=\dfrac{WR}{\pi}$

At $\theta=\dfrac{\pi}{2},\; M_1=-WR\left(\dfrac{1}{2}-\dfrac{1}{\pi}\right)$

Stress at $A=\dfrac{M_0}{R(A'-A)}\left\{\dfrac{A'}{A}-\dfrac{R}{R+r}\right\}$

$$=\frac{W}{\pi\times 0{\cdot}016\pi}\left\{1{\cdot}016-\frac{4}{5}\right\}=1{\cdot}368W \text{ (tensile)}$$

$$\text{Stress at } B = \frac{M_0}{R(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R-r}\right\}$$

$$= \frac{W}{\pi \times 0{\cdot}016\pi}\left\{1{\cdot}016 - \frac{4}{3}\right\} = -2{\cdot}007W \text{ (compressive)}$$

$$\text{Stress at } C = \frac{M_1}{R(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R-r}\right\} + \frac{W}{2A}$$

$$= \frac{-W\left(\frac{1}{2} - \frac{1}{\pi}\right)}{0{\cdot}016\pi}\left\{1{\cdot}016 - \frac{4}{3}\right\} + \frac{W}{2\pi}$$

$$= 1{\cdot}146W + 0{\cdot}159W = 1{\cdot}305W \text{ (tensile)}$$

$$\text{Stress at } D = \frac{M_1}{R(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R+r}\right\} + \frac{W}{2A}$$

$$= \frac{-W\left(\frac{1}{2} - \frac{1}{\pi}\right)}{0{\cdot}016\pi}\left\{1{\cdot}016 - \frac{4}{5}\right\} + \frac{W}{2\pi}$$

$$= -0{\cdot}781W + 0{\cdot}159W = -0{\cdot}622W \text{ (comp.)}$$

The stresses at A, B, C and D due to the horizontal load are similarly known.

They are tabulated as below.

Point	Stress due to vertical load	Stress due to horizontal load	Total stress
A	$+1{\cdot}368W$	$-0{\cdot}622W$	$+0{\cdot}746W$
B	$-2{\cdot}007W$	$+1{\cdot}305W$	$-0{\cdot}702W$
C	$+1{\cdot}305W$	$-2{\cdot}007W$	$-0{\cdot}702W$
D	$-0{\cdot}622W$	$+1{\cdot}368W$	$+0{\cdot}746W$

Maximum tensile stress at A or $D = 0{\cdot}746W$

Hence $0{\cdot}746W = 800$

$\therefore$ **W = 1,072 kg.**

14. Fig. 36 shows a circular ring of mean radius R, made of material having a uniform section. The ring is fitted with a rigid bar across the diameter AB and transmits a pull W along the diameter CD which is at right angles to AB. Obtain the force in the bar in terms of W, and the change in the diameter CD in terms of W and the flexural rigidity EI of the ring. (Lond. Univ.)

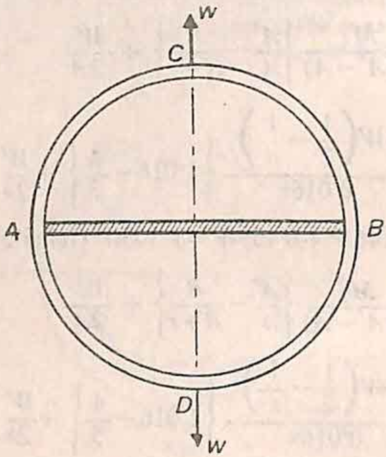

Fig. 36

Let P be the force in the rigid bar AB.

Change of diameter along AB,

$$0=0{\cdot}149\frac{PR^3}{EI}-0{\cdot}137\frac{WR^3}{EI}$$

$$\therefore\quad P=0{\cdot}919W$$

Change of diameter along CD,

$$=0{\cdot}149\frac{WR^3}{EI}-0{\cdot}137\frac{PR^3}{EI}$$

$$=(0{\cdot}149-0{\cdot}137\times0{\cdot}919)\frac{WR^3}{EI}$$

$$=0{\cdot}0231\frac{WR^3}{EI}\text{ increase.}$$

15. A proving ring is 25 cm mean diameter, 4 cm wide and 6 mm thick. The maximum stress permitted is 5,500 kg/cm². Find the load to cause this stress, and the load to give 6 mm deflection of the ring in the direction of the loading. $E=2\times10^6$ kg/cm². (Lond. Univ.)

$$M=WR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$$

Maximum M occurs at $\theta=0$ and is given by

$$M_0=\frac{WR}{\pi}$$

Maximum negative M occurs at $\theta=\frac{\pi}{2}$ and is given by

$$M_1=-WR\left(\frac{1}{2}-\frac{1}{\pi}\right)$$

In this case the radius of curvature is large compared with the dimensions of the cross-section and the analysis of stress is similar to that of simple bending.

Along the load line maximum stress,

$$f=\frac{M_0}{Z}=\frac{WR}{\pi Z}$$

$$Z=\frac{1}{6}\times 4\times(0{\cdot}6)^2=\frac{6}{25}\text{ cm}^3$$

$$\therefore\quad f=\frac{W\times 12{\cdot}5\times 25}{\pi\times 6}=16{\cdot}58W$$

Perpendicular to the load line maximum tensile stress,

$$f=\frac{M_1}{Z}+\frac{W}{2A}=\frac{WR\left(\frac{1}{2}-\frac{1}{\pi}\right)}{Z}+\frac{W}{2A}$$

$$=\frac{W\times 12{\cdot}5\left(\frac{1}{2}-\frac{1}{\pi}\right)\times 25}{6}+\frac{W}{2\times 4\times 0{\cdot}6}$$

$$=9{\cdot}67W$$

Hence $\quad 16{\cdot}58W=5{,}500$

$$\therefore\quad \mathbf{W=332\ kg}$$

Deflection in the direction of loading

$$0{\cdot}6=0{\cdot}149\frac{WR^3}{EI}$$

$$=\frac{0{\cdot}149\times W\times 12{\cdot}5^3}{2\times 10^6\times \frac{1}{12}\times 4\times(0{\cdot}6)^3}$$

$$\therefore\quad W=\mathbf{297\ kg.}$$

16. The links of a chain are made of 2 cm round steel and have semi-circular ends, the mean radius of which is 2·5 cm. The ends are connected by straight pieces 2 cm long. Estimate the greatest intensities of tensile and compressive stress in the link when the chain sustains a load of 600 kg.

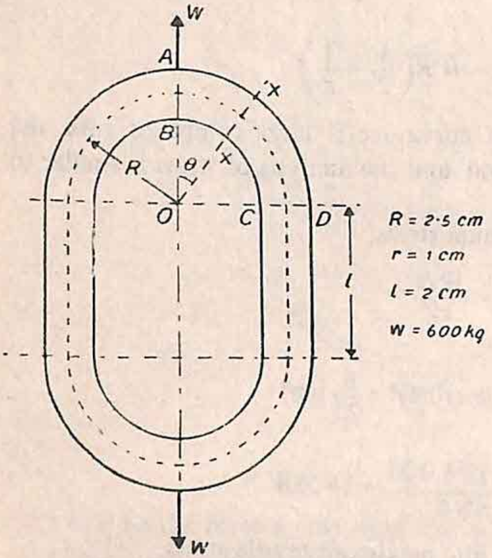

Fig. 37

$$A = \pi \text{ cm}^2$$

$$A' = 2\pi \times 2{\cdot}5(2{\cdot}5 - \sqrt{6{\cdot}25 - 1})$$

$$= 5\pi(2{\cdot}5 - 2{\cdot}2913)$$

$$= 1{\cdot}0435\pi \text{ cm}^2$$

$$A' - A = 0{\cdot}0435\pi,$$

$$\frac{A'}{A} = 1{\cdot}0435$$

$$M = \frac{WR}{2}\left(\frac{2R+l}{\pi R+l} - \sin\theta\right)$$

At AB where $\theta = 0$,

$$M = M_0 = \frac{WR}{2} \cdot \frac{2R+l}{\pi R+l}$$

At CD where $\theta = \frac{\pi}{2}$

$$M = M_1 = -\frac{WR}{2} \cdot \frac{\pi R - 2R}{\pi R + l}$$

Stress at A (extrados, along the load line)

$$= \frac{M_0}{R(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R+r}\right\}$$

$$= \frac{W(2R+l)}{2(\pi R+l)(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R+r}\right\}$$

$$= \frac{600 \times 7}{2(2{\cdot}5\pi + 2) \times 0{\cdot}0435\pi}\left\{1{\cdot}0435 - \frac{2{\cdot}5}{3{\cdot}5}\right\}$$

$$= 513 \text{ kg/cm}^2 \text{ (tension)}$$

Stress at B (intrados, along the load line)

$$= \frac{M_0}{R(A'-A)}\left\{\frac{A'}{A} - \frac{R}{R-r}\right\}$$

$$=\frac{600\times 7}{2(2\cdot5\pi+2)\times 0\cdot0435\pi}\left\{1\cdot0435-\frac{2\cdot5}{1\cdot5}\right\}$$

$$=-972 \text{ kg/cm}^2 \text{ (comp.)}$$

Stress at C (intrados, perpendicular to the load line)

$$=\frac{M_1}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}+\frac{W}{2A}$$

$$=\frac{-W(\pi R-2R)}{2(\pi R+l)(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R-r}\right\}+\frac{W}{2A}$$

$$=\frac{-600\times 2\cdot5(\pi-2)}{2(2\cdot5\pi+2)\times 0\cdot0435\pi}\left\{1\cdot0435-\frac{2\cdot5}{1\cdot5}\right\}+\frac{600}{2\pi}$$

$$=396+95=491 \text{ kg/cm}^2 \text{ (tension)}$$

Stress at D (extrados, perpendicular to the load line)

$$=\frac{M_1}{R(A'-A)}\left\{\frac{A'}{A}-\frac{R}{R+r}\right\}+\frac{W}{2A}$$

$$=\frac{-600\times 2\cdot5(\pi-2)}{2(2\cdot5\pi+2)\times 0\cdot0435\pi}\left\{1\cdot0435-\frac{2\cdot5}{3\cdot5}\right\}+\frac{600}{2\pi}$$

$$=-209+95=-114 \text{ kg/cm}^2 \text{ (comp.)}$$

Maximum tension at $A=$ **513 kg/cm²**

Maximum compression at $B=$ **−972 kg/cm².**

17. The links of a chain are made of circular section rod 6 mm diameter and have semi-circular ends, the mean radius of which is 25 mm. The ends are connected by straight pieces 45 mm long. Calculate the ratio of the maximum tensile stress at the section where the load is applied to that at the section half-way along the straight portion.

$$M=\frac{WR}{2}\left(\frac{2R+l}{\pi R+l}-\sin\theta\right)$$

At $\quad \theta=0,\ M_0=\dfrac{WR}{2}\cdot\dfrac{2R+l}{\pi R+l}$

At $\quad \theta=\dfrac{\pi}{2},\ M_1=-\dfrac{WR}{2}\cdot\dfrac{\pi R-2R}{\pi R+l}$

Here R is large compared with d and the theory of simple bending will apply.

Maximum tensile stress at extrados along the load line

$$=\frac{M_0}{Z}=\frac{WR}{2}\cdot\frac{2R+l}{\pi R+l}\cdot\frac{32}{\pi d^3}$$

$$=\frac{W\times 2\cdot5}{2}\times\frac{5+4\cdot5}{2\cdot5\pi+4\cdot5}\times\frac{32}{\pi\times(0\cdot6)^3}$$

$$=45\cdot3\ W$$

Maximum tensile stress at the inner side of the straight portion

$$=\frac{M_1}{Z}+\frac{W}{2A}$$

$$=\frac{WR}{2}\cdot\frac{\pi R-2R}{\pi R+l}\cdot\frac{32}{\pi d^3}+\frac{W\times4}{2\times\pi d^2}$$

$$=\frac{W\times2\cdot5}{2}\times\frac{2\cdot5(\pi-2)}{2\cdot5\pi+4\cdot5}\times\frac{32}{\pi\times(0\cdot6)^3}+\frac{2W}{\pi\times(0\cdot6)^2}$$

$$=13\cdot61W+1\cdot77W=15\cdot38W$$

Ratio $$=\frac{45\cdot3W}{15\cdot38W}=2\cdot95.$$

18. A steel ring of rectangular cross-sectien 8 mm wide by 5 mm thick has a mean diameter of 30 cm. A narrow radial saw cut is made, and tangential separating forces of $\frac{1}{2}$ kg each are applied at the cut in the plane of the ring. Find the additional separation due to these forces. $E=2\times10^6$ kg/cm². (Lond. Univ.)

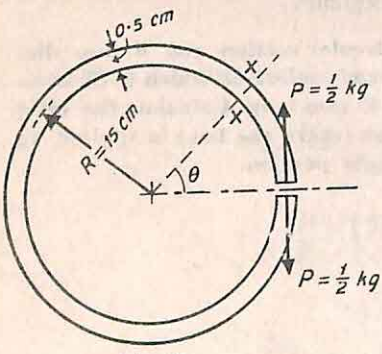

Fig. 38

$$U=\frac{1}{2EI}\int M^2\,ds$$

$$=\frac{2}{2EI}\int_0^\pi M^2.R\,d\theta$$

$$=\frac{R}{EI}\int_0^\pi M^2 d\theta$$

Additional separation, $\delta=\frac{\partial U}{\partial P}=\frac{2R}{EI}\int_0^\pi M\frac{\partial M}{\partial P}.d\theta$

M at $XX = -PR(1-\cos\theta)$

$$\frac{\partial M}{\partial P} = -R(1-\cos\theta)$$

$$\therefore \quad \delta = \frac{2R}{EI}\int_0^{\pi} PR^2(1-\cos\theta)^2 d\theta$$

$$= \frac{2PR^3}{EI}\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$= \frac{2PR^3}{EI}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \frac{2PR^3}{EI}\left(\pi + \frac{\pi}{2}\right) = \frac{3\pi PR^3}{EI}$$

$$= \frac{3\pi \times \frac{1}{2} \times 15^3}{2\times 10^6 \times \frac{1}{12} \times 0{\cdot}8 \times (0{\cdot}5)^3} = 0{\cdot}954 \text{ cm.}$$

19. A steel spring ABC, of the dimensions shown in Fig. 39 is firmly clamped at A. If a load of 2 kg is supported at C, find the vertical deflection at this point. Neglect the effect of shear and take $E = 2\times 10^6$ kg/cm².
(Lond. Univ.)

Portion AB.

$$M = P(12 + 6\sin\theta)$$

$$\frac{\partial M}{\partial P} = 12 + 6\sin\theta$$

Portion BC.

$$M = Px$$

$$\frac{\partial M}{\partial P} = x$$

$$U = \int \frac{M^2 ds}{2EI}$$

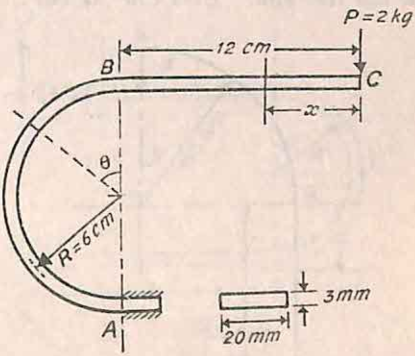

Fig. 39

Vertical deflection at C

$$= \frac{\partial U}{\partial P} = \frac{1}{EI}\int M\frac{\partial M}{\partial P} ds$$

$$=\frac{1}{EI}\left[\int_0^{\pi} P(12+6\sin\theta)^2\times 6\,d\theta+\int_0^{12} Px^2\,dx\right]$$

$$=\frac{1}{EI}\left\{6P\int_0^{\pi}(144+144\sin\theta+36\sin^2\theta)d\theta+\frac{P}{3}\left(x^3\right)_0^{12}\right\}$$

$$=\frac{1}{EI}\left\{6P\left(144\theta-144\cos\theta+18\theta-9\sin 2\theta\right)_0^{\pi}+576P\right\}$$

$$=\frac{1}{EI}\left\{6P\left(144\pi+288+18\pi\right)+576P\right\}$$

$$=\frac{P}{EI}(972\pi+2{,}304)$$

$$=\frac{2(972\pi+2{,}304)}{2\times 10^6\times\frac{1}{12}\times 2\times(0{\cdot}3)^3}$$

$$=\mathbf{1{\cdot}19\ cm.}$$

20. A steel bar 6 cm diameter is bent to the shape shown in Fig. 40 and the lower end is firmly fixed in the ground in a vertical position. A load of 80 kg is applied at the free end. Calculate the vertical deflection of the free end. $E=2\times10^6$ kg/cm². (Lond. Univ.)

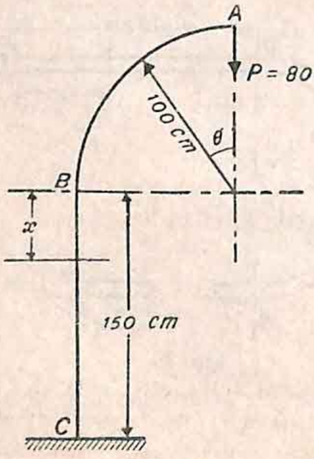

Fig. 40

Quadrant AB.

$$M=P\times 100\sin\theta$$

$$\frac{\partial M}{\partial P}=100\sin\theta$$

Length BC.

$$M=100P$$

$$\frac{\partial M}{\partial P}=100$$

Vertical deflection at A

$$=\frac{1}{EI}\int M\frac{\partial M}{\partial P}\cdot ds$$

$$=\frac{1}{EI}\left[\int_0^{\pi/2} 100P\sin\theta.100\sin\theta.100\,d\theta+\int_0^{150} 100P.100.dx\right]$$

$$=\frac{1}{EI}\left[100^3\times P\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)_0^{\pi/2}+100^2\times P\left(x\right)_0^{150}\right]$$

$$=\frac{1}{EI}\left[100^3\times P\times\frac{\pi}{4}+100^2\times P\times 150\right]$$

$$=\frac{P\times 100^2}{EI}(25\pi+150)$$

$$=\frac{80\times 100^2\times(25\pi+150)\times 64}{2\times 10^6\times\pi\times 6^4}$$

$$=\mathbf{1{\cdot}437\ cm.}$$

21. A spring used in a measuring device is made of a rod of steel of diameter d bent to the form shown in Fig. 41 so that a force P applied to the ends of the spring will increase the distance between the ends by an amount δ. Show that the stiffness of the spring

$$S=\frac{P}{\delta}=\frac{3\pi Ed^4}{32}\Big/(4L^3+6\pi RL^2+24R^2L+3\pi R^3)$$

If S is to be $1{\cdot}5$ kg/cm, $d=0{\cdot}6$ cm, $R=4$ cm, find the length L. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

Portion AB.

$$M=Px$$

$$\frac{\partial M}{\partial P}=x$$

Quadrant BC.

$$M=P(L+R\sin\theta)$$

$$\frac{\partial M}{\partial P}=L+R\sin\theta$$

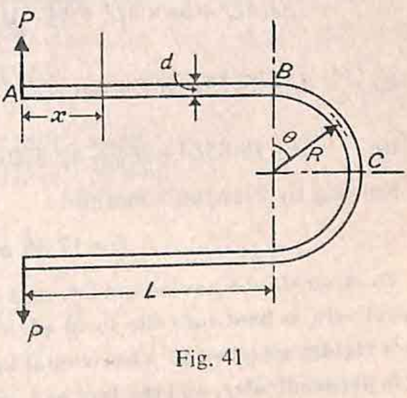

Fig. 41

$$U=\frac{1}{2EI}\int M^2 ds$$

$$\delta=\frac{\partial U}{\partial P}=\frac{1}{EI}\int M.\frac{\partial M}{\partial P}.ds$$

$$=\frac{2}{EI}\left[\int_0^L Px^2dx+\int_0^{\pi/2} P(L+R\sin\theta)^2R\,d\theta\right]$$

$$=\frac{2}{EI}\left[\frac{PL^3}{3}+PR\int_0^{\pi/2}(L^2+2RL\sin\theta+R^2\sin^2\theta)d\theta\right]$$

$$=\frac{2}{EI}\left[\frac{PL^3}{3}+PR\left\{L^2\theta-2RL\cos\theta+R^2\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)\right\}_0^{\pi/2}\right]$$

$$=\frac{2}{EI}\left[\frac{PL^3}{3}+PR\left(L^2.\frac{\pi}{2}+2RL+R^2.\frac{\pi}{4}\right)\right]$$

$$=\frac{P}{6EI}(4L^3+6\pi RL^2+24R^2L+3\pi R^3)$$

$$\therefore\quad S=\frac{P}{\delta}=\frac{6E.\frac{\pi}{64}d^4}{4L^3+6\pi RL^2+24R^2L+3\pi R^3}$$

$$=\frac{3\pi Ed^4}{32}\Big/(4L^3+6\pi RL^2+24R^2L+3\pi R^3)$$

$$1{\cdot}5=\frac{3\pi\times2\times10^6\times(0{\cdot}6)^4}{32(4L^3+6\pi\times4L^2+24\times16L+3\pi\times64)}$$

or $$L^3+6\pi L^2+96L+48\pi=\frac{3\pi\times2\times10^6\times(0{\cdot}6)^4}{1{\cdot}5\times32\times4}$$

or $$L^3+18{\cdot}85L^2+96L-12{,}570=0$$

Solving by Newton's method

$$L=17{\cdot}35\text{ cm.}$$

22. A steel tube having outside and inside diameters of 4 cm and 3 cm respectively, is bent into the form of a quadrant of 1·5 m radius. One end is rigidly attached to a horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 20 kg. Determine the vertical and horizontal deflections of the free end under this load. $E=2\times10^6$ kg/cm².

(Lond. Univ.)

$$I=\frac{\pi}{64}(4^4-3^4)=\frac{175\pi}{64}\text{ cm}^4$$

Apply a fictitious load Q as shown in figure.

$$M=PR\sin\theta+QR(1-\cos\theta)$$

$$\frac{\partial M}{\partial P}=R\sin\theta$$

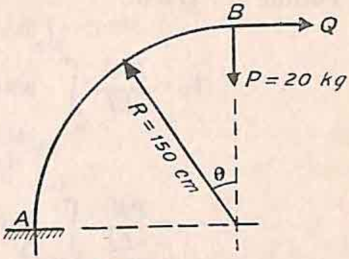

Fig. 42

$$\frac{\partial M}{\partial Q}=R(1-\cos\theta)$$

Vertical deflection at B

$$\delta_V=\frac{\partial U}{\partial P}=\frac{1}{EI}\int_0^{\pi/2} M.\frac{\partial M}{\partial P}\,.\,ds$$

$$=\frac{1}{EI}\int_0^{\pi/2}\{PR\sin\theta+QR(1-\cos\theta)\}\times R\sin\theta.Rd\theta$$

Putting $Q=0$,

$$\delta_V=\frac{PR^3}{EI}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$=\frac{PR^3}{EI}\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)_0^{\pi/2}$$

$$=\frac{PR^3}{EI}\cdot\frac{\pi}{4}=\frac{20\times150^3\times\pi\times64}{2\times10^6\times175\pi\times4}$$

$$=3{\cdot}09\text{ cm}$$

Horizontal deflection at B

$$\delta_H=\frac{\delta U}{\delta Q}=\frac{1}{EI}\int M\frac{\partial M}{\partial Q}ds$$

$$=\frac{1}{EI}\int_0^{\pi/2}\{PR\sin\theta+QR(1-\cos\theta)\}R(1-\cos\theta)R\,d\theta$$

Putting $Q=0$

$$\delta_H=\frac{PR^3}{EI}\int_0^{\pi/2}\sin\theta(1-\cos\theta)d\theta$$

$$=\frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$=\frac{PR^3}{EI}\left[-\cos\theta+\frac{\cos 2\theta}{4}\right]_0^{\pi/2}$$

$$=\frac{PR^3}{EI}\left[1+\frac{1}{4}(-1-1)\right]$$

$$=\frac{PR^3}{2EI}=\frac{20\times 150^3\times 64}{2\times 2\times 10^6\times 175\pi}$$

1·964 cm.

23. A strip of steel of rectangular section 25 mm$\times$3 mm is bent to the shape of a quadrant and loaded by a force W inclined at α to the vertical as shown in Fig. 43. Derive formulae for the vertical and horizontal movements at the free end B and hence find the value of α to give no horizontal movement at B.

For the value of α found, determine the radius R to give a vertical deflection at B of 0·025 cm when $W=\frac{1}{2}$ kg. $E=2\times 10^6$ kg/cm^2. (Lond. Univ.)

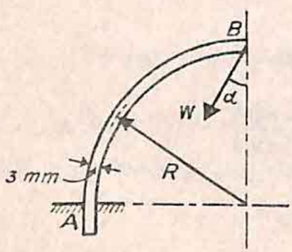

Fig. 43

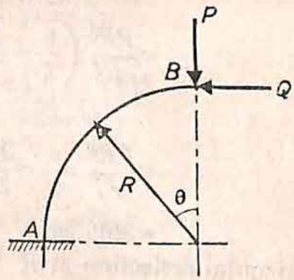

Fig. 44

The given force W is replaced by its vertical and horizontal components P and Q as shown in Fig. 44, where

$$P=W\cos\alpha,\ Q=W\sin\alpha$$

$$M=PR\sin\theta-QR(1-\cos\theta)$$

$$\frac{\partial M}{\partial P}=R\sin\theta,\ \frac{\partial M}{\partial Q}=-R(1-\cos\theta)$$

Vertical deflection at B,

$$\delta_V=\frac{\partial U}{\partial P}=\frac{1}{EI}\int M\frac{\partial M}{\partial P}ds$$

$$=\frac{1}{EI}\int_0^{\pi/2}\{PR\sin\theta-QR(1-\cos\theta)\}R\sin\theta.R\,d\theta$$

$$=\frac{R^3}{EI}\int_0^{\pi/2}\left(P\sin^2\theta-Q\sin\theta+\frac{Q}{2}\sin 2\theta\right)d\theta$$

$$=\frac{R^3}{EI}\left[P\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)+Q\cos\theta-\frac{Q}{4}\cos 2\theta\right]_0^{\pi/2}$$

$$=\frac{R^3}{EI}\left[P.\frac{\pi}{4}-Q+\frac{Q}{2}\right]$$

$$=\mathbf{\frac{R^3}{4EI}(\pi P-2Q)}$$

Horizontal deflection at B,

$$\delta_H=\frac{\partial U}{\partial Q}=\frac{1}{EI}\int M\frac{\partial M}{\partial Q}ds$$

$$=\frac{1}{EI}\int_0^{\pi/2}\{PR\sin\theta-QR(1-\cos\theta)\}\times\{-R(1-\cos\theta)\}R\,d\theta$$

$$=\frac{R^3}{EI}\int_0^{\pi/2}\left[-P\sin\theta+\frac{P}{2}\sin 2\theta+Q(1-\cos\theta)^2\right]d\theta$$

$$=\frac{R^3}{EI}\left[P\cos\theta-\frac{P}{4}\cos 2\theta+Q\left(\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]^{\pi/2}$$

$$=\frac{R^3}{EI}\left[-P+\frac{P}{2}+Q\left(\frac{3}{2}\times\frac{\pi}{2}-2\right)\right]$$

$$=\frac{R^3}{4EI}[(3\pi-8)Q-2P]$$

For horizontal deflection at B to be zero

$$(3\pi-8)Q-2P=0$$

$$\therefore \quad \tan\alpha=\frac{Q}{P}=\frac{2}{3\pi-8}=1{\cdot}404$$

$$\alpha=54^\circ{\cdot}32'$$

$$P=W\cos\alpha=\tfrac{1}{2}\times\cos 54^\circ\ 32'=0{\cdot}290 \text{ kg}$$

$$Q=W\sin\alpha=\tfrac{1}{2}\times\sin 54^\circ\ 32'=0{\cdot}407 \text{ kg}$$

$$0{\cdot}025=\frac{R^3(\pi\times 0{\cdot}290-2\times 0{\cdot}407)}{4\times 2\times 10^6\times\frac{1}{12}\times 2{\cdot}5\times(0{\cdot}3)^3}$$

$$\therefore \quad R=22{\cdot}6 \text{ cm.}$$

24. A steel spring is shown in Fig. 45. It is formed in a circular arc subtending an angle of 270° at the centre. The lower end is rigidly fixed and a vertical force of $\frac{1}{2}$ kg is applied at the free end.

If the section of the spring is 12 mm wide and 3 mm thick, calculate the vertical and horizontal displacements of the free end. $E=2\times 10^6$ kg/cm².

(Lond. Univ.)

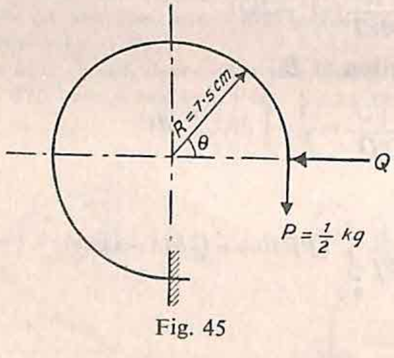

Fig. 45

Apply a fictitious load Q as shown in Fig. 45.

$$M=PR(1-\cos\theta)+QR\sin\theta$$

$$\frac{\partial M}{\partial P}=R(1-\cos\theta)$$

$$\frac{\partial M}{\partial Q}=R\sin\theta$$

Vertical deflection at the free end

$$\frac{\partial U}{\partial P}=\frac{1}{EI}\int M\frac{\partial M}{\partial P}\,ds$$

$$=\frac{1}{EI}\int_0^{3\pi/2} PR(1-\cos\theta).R(1-\cos\theta).R\,d\theta$$

(since $Q=0$)

$$=\frac{PR^3}{EI}\int_0^{3\pi/2}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$=\frac{PR^3}{EI}\left[\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{3\pi/2}$$

$$=\frac{PR^3}{EI}\left[\frac{3}{2}.\frac{3\pi}{2}+2\right]=\frac{PR^3}{EI}\left(\frac{9\pi}{4}+2\right)$$

$$=\frac{7{\cdot}5^3\times\left(\frac{9\pi}{4}+2\right)}{2\times2\times10^6\times\frac{1}{12}\times1{\cdot}2\times(0{\cdot}3)^3}$$

$=\mathbf{0{\cdot}354\ cm}$

Horizontal deflection at the free end

$$\frac{\partial U}{\partial Q}=\frac{1}{EI}\int M\frac{\partial M}{\partial Q}\,ds$$

$$=\frac{1}{EI}\int_0^{3\pi/2} PR(1-\cos\theta).R\sin\theta.R d\theta$$

(since $Q=0$)

$$=\frac{PR^3}{EI}\int_0^{3\pi/2}\left(\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$=\frac{PR^3}{EI}\left[-\cos\theta+\frac{\cos 2\theta}{4}\right]_0^{3\pi/2}$$

$$= \frac{PR^3}{EI}\left[1-\frac{1}{2}\right] = \frac{PR^3}{2EI}$$

$$= \frac{7 \cdot 5^3}{2\times 2\times 2\times 10^6 \times \frac{1}{12} \times 1 \cdot 2 \times (0 \cdot 3)^3}$$

$$= \mathbf{0 \cdot 0195 \text{ cm.}}$$

25. Fig. 46 shows a steel rod 1 cm diameter with one end firmly fixed into a horizontal table. The remainder of the rod is bent into the form of three-quarters of a circle and the free end of the rod is constrained by guides to move vertically. If the mean radius to which the rod is bent is 15 cm, determine the vertical deflection of the free end for a load of 10 kg gradually applied there. $E = 2\times 10^6$ kg/cm². (Lond. Univ.)

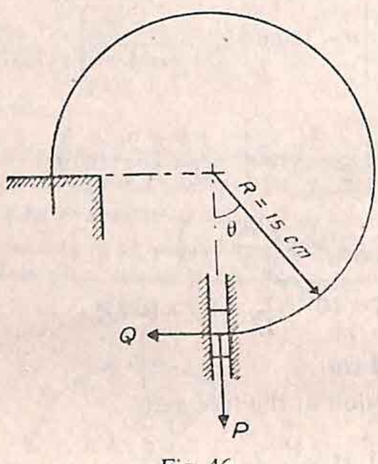

Fig. 46

Let the vertical load be P and the normal reaction due to the constraint be Q.

Then $M = Q\times 15(1-\cos\theta) - P\times 15\sin\theta$

$$\frac{\partial M}{\partial Q} = 15(1-\cos\theta)$$

$$\frac{\partial M}{\partial P} = -15\sin\theta$$

Since there is no horizontal displacement

$$\frac{\partial U}{\partial Q} = 0, \quad \text{i.e.,} \int M\frac{\partial M}{\partial Q}\, ds = 0$$

or $$\int_0^{3\pi/2} \{15Q(1-\cos\theta)-15P\sin\theta\}15(1-\cos\theta).15\, d\theta=0$$

or $$\int_0^{3\pi/2} \{Q(1-\cos\theta)-P\sin\theta\}(1-\cos\theta)d\theta=0$$

or $$\int_0^{3\pi/2} \left\{Q(1-2\cos\theta+\cos^2\theta)-P\left(\sin\theta-\frac{\sin2\theta}{2}\right)\right\} d\theta=0$$

or $$\left[Q\left(\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin2\theta}{4}\right)-P\left(-\cos\theta+\frac{\cos2\theta}{4}\right)\right]_0^{3\pi/2}=0$$

or $$Q\left(\frac{3}{2}\cdot\frac{3\pi}{2}+2\right)-P\left(1-\frac{1}{2}\right)=0$$

or $$Q=\frac{2P}{9\pi+8}=\frac{20}{9\pi+8}=0{\cdot}551 \text{ kg}$$

Vertical displacement

$$=\frac{\partial U}{\partial P}=\frac{1}{EI}\int M\frac{\partial M}{\partial P}\, ds$$

$$=\frac{15^3}{EI}\int_0^{3\pi/2} \{Q(1-\cos\theta)-P\sin\theta\}(-\sin\theta)d\theta$$

$$=\frac{15^3}{EI}\int_0^{3\pi/2} \left\{Q\left(-\sin\theta+\frac{\sin2\theta}{2}\right)+P\sin^2\theta\right\} d\theta$$

$$=\frac{15^3}{EI}\left[Q\left(\cos\theta-\frac{\cos2\theta}{4}\right)+P\left(\frac{\theta}{2}-\frac{\sin2\theta}{4}\right)\right]_0^{3\pi/2}$$

$$=\frac{15^3}{EI}\left[Q\left(-1+\frac{1}{2}\right)+P\left(\frac{3\pi}{4}\right)\right]$$

$$=\frac{15^3}{EI}\left(-\frac{Q}{2}+\frac{3\pi}{4}P\right)$$

$$=\frac{15^3\times 64}{2\times 10^6\times\pi}\left(-\frac{0{\cdot}551}{2}+\frac{3\pi}{4}\times 10\right)$$

$$=\mathbf{0{\cdot}801\ cm.}$$

26. The ring shown in Fig. 47 is made of flat steel strip 20 mm $\times$ 3 mm and is shaped in the form of a circle of mean diameter 20 cm. The ends at B are cut square and not joined. A pull P is applied along the diameter CD which is at right angles to the diameter AB. If the maximum tensile stress due to P is 1,250 kg/cm², find the increase in the opening at B. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

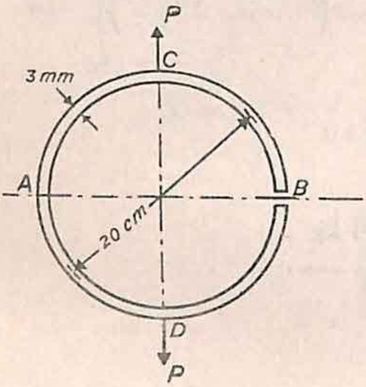

Fig. 47

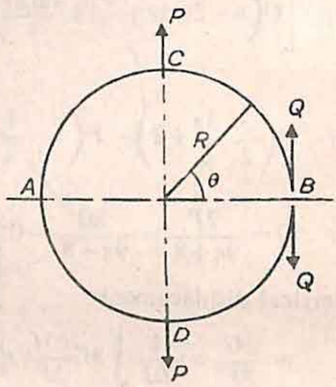

Fig. 48

Fictitious forces Q are applied at B as shown in Fig. 48.

For BC, $$M=-QR(1-\cos\theta)$$

$$\frac{\partial M}{\partial Q}=-R(1-\cos\theta)$$

For CA, $$M=-QR(1-\cos\theta)+PR\cos\theta$$

$$\frac{\partial M}{\partial Q}=-R(1-\cos\theta)$$

Increase in the opening at B,

$$\delta=\frac{\partial U}{\partial Q}=\frac{1}{EI}\int M\frac{\partial M}{\partial Q}\,ds$$

$$=\frac{2}{EI}\left[\int_0^{\pi/2} QR^2(1-\cos\theta)^2 R\,d\theta+\int_{\pi/2}^{\pi}\{-QR(1-\cos\theta)\right.$$

$$\left.+PR\cos\theta\}\times\{-R(1-\cos\theta)\}R\,d\theta\right]$$

Putting $\quad Q=0$

$$\delta=\frac{2}{EI}\int_{\pi/}^{\pi} -PR^3\cos\theta(1-\cos\theta)d\theta$$

$$=-\frac{2PR^3}{EI}\left[\sin\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

$$=-\frac{2PR^3}{EI}\left[-1-\frac{\pi}{4}\right]$$

$$=\frac{2PR^3}{EI}\left(1+\frac{\pi}{4}\right)$$

The maximum stress occurs at A, where

$$M=-PR=-10P$$

$$Z=\tfrac{1}{6}\times 2\times(0{\cdot}3)^2=0{\cdot}03\text{ cm}^3$$

$$f=\frac{M}{Z}$$

i.e., $\quad 1{,}250=\dfrac{10P}{0{\cdot}03}$

$$\therefore\quad P=3{\cdot}75\text{ kg}$$

$$\therefore\quad \delta=\frac{2\times 3{\cdot}75\times 1{,}000}{2\times 10^6\times\frac{1}{12}\times 2\times(0{\cdot}3)^3}\times\left(1+\frac{\pi}{4}\right)$$

$$=1{\cdot}488\text{ cm.}$$

27. A ring of mean radius R is made from a bar of uniform section with the two ends at C connected by a pin-joint. The ring is subjected to three radial forces arranged in equilibrium as shown in Fig. 49. Show that the force in the pin is

(Lond. Univ.)

$$F=\frac{2P}{3\pi}\{8+2(\pi-\alpha)\}$$

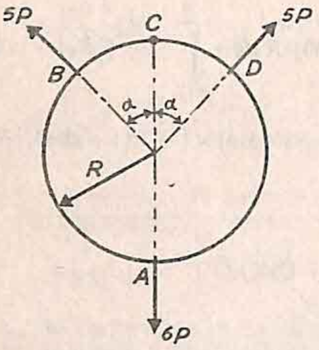

Fig. 49

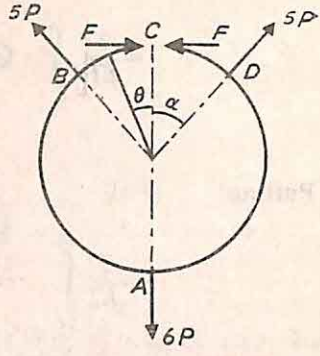

Fig. 50

For equilibrium, $2 \times 5P\cos\alpha = 6P$

$\therefore \quad \cos\alpha = \frac{3}{5},\ \sin\alpha = \frac{4}{5}$

Let F be the force at the pin as shown in Fig. 50.

Then $$\frac{\partial U}{\partial F} = 0$$

i.e., $$\int M\frac{\partial M}{\partial F}\,ds = 0$$

For BC, $M = FR(1-\cos\theta)$

$$\frac{\partial M}{\partial F} = R(1-\cos\theta)$$

For AB, $M = FR(1-\cos\theta) - 5PR\sin(\theta-\alpha)$

$$\frac{\partial M}{\partial F} = R(1-\cos\theta)$$

Hence $$2\int_0^\alpha FR^3(1-\cos\theta)^2\,d\theta + 2\int_\alpha^\pi \{FR(1-\cos\theta) - 5PR\sin(\theta-\alpha)\}\,R(1-\cos\theta)R\,d\theta = 0$$

or $$\int_0^\alpha F(1-\cos\theta)^2\,d\theta + \int_\alpha^\pi F(1-\cos\theta)^2 d\theta - \int_\alpha^\pi 5P\sin(\theta-\alpha)(1-\cos\theta)d\theta = 0$$

or $$F\int_0^\pi (1-\cos\theta)^2 d\theta = 5P\int_\alpha^\pi \{\sin(\theta-\alpha)-\sin(\theta-\alpha)\cos\theta\}d\theta$$

or $$F\int_0^\pi (1-2\cos\theta+\cos^2\theta)d\theta = 5P\int_\alpha^\pi \{\sin(\theta-\alpha)-\frac{1}{2}\sin(2\theta-\alpha)$$
$$+\tfrac{1}{2}\sin\alpha\}d\theta$$

or $$F\left[\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^\pi$$

$$=5P\left[-\cos(\theta-\alpha)+\frac{1}{4}\cos(2\theta-\alpha)+\frac{\theta}{2}\sin\alpha\right]_\alpha^\pi$$

or $$F\left(\frac{3}{2}\pi\right)=5P\left(\cos\alpha+1+\frac{\pi-\alpha}{2}\sin\alpha\right)$$
$$=5P\left(\frac{3}{5}+1+\frac{\pi-\alpha}{2}\times\frac{4}{5}\right)$$
$$=P\{8+2(\pi-\alpha)\}$$
$$\therefore\quad F=\frac{2P}{3\pi}\{8+2(\pi-\alpha)\}$$

CHAPTER VII

UNSYMMETRICAL BENDING

1. A steel bar, of rectangular section 6×3 cm, is arranged as a cantilever projecting horizontally 60 cm beyond the support. The broad face of the bar makes 30° with the horizontal. A load of 10 kg is hung from the free end. Find the neutral axis, the horizontal and vertical deflections of the free end, and the maximum tensile stress. $E=2\times 10^6$ kg/cm².

(Lond. Univ.)

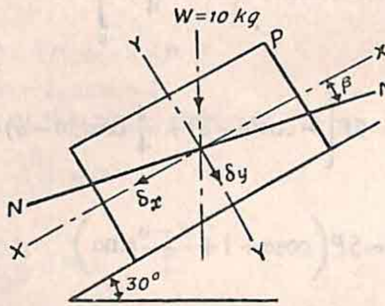

Fig. 51

$$I_x=\frac{6\times 3^3}{12}=13{\cdot}5 \text{ cm}^4$$

$$I_y=\frac{3\times 6^3}{12}=54 \text{ cm}^4$$

At the fixed end $M=10\times 60=600$ kg cm acting in a vertical plane.

Component in the plane YY is

$M_x=M\cos 30^\circ$, causing bending about XX.

Component in the plane XX is

$M_y=M\sin 30^\circ$, causing bending about YY.

$$\text{Stress due to } M_x=\frac{M_x}{I_x}\,.\,y=\frac{M\cos 30^\circ}{I_x}\quad y$$

$$\text{Stress due to } M_y=\frac{M_y}{I_y}\,.\,x=\frac{M\sin 30^\circ}{I_y}\quad x$$

$$\text{Total stress}\qquad f=\frac{M\cos 30^\circ}{I_x}\,y+\frac{M\sin 30^\circ}{I_y}\,x$$

At the neutral axis $f=0$, so that

$$y=-\frac{I_x}{I_y}\tan 30^\circ . x$$

If β is the inclination of the neutral axis with the XX axis, then

$$\tan\beta=-\frac{I_x}{I_y}\tan 30^\circ=-\frac{13\cdot 5}{54}\tan 30^\circ=-0\cdot 1444$$

$$\therefore \quad \beta=-8^\circ 13'$$

Hence inclination to horizontal $=30^\circ-8^\circ 13'=$ **21°47′**. The neutral axis NN is shown in the figure.

Maximum tensile stress at P

$$=600\left(\frac{1\cdot 5\cos 30^\circ}{13\cdot 5}+\frac{3\sin 30^\circ}{54}\right)=\mathbf{74\cdot 4\ kg/cm^2}$$

Deflection along YY, $\delta_y=\dfrac{W\cos 30^\circ . L^3}{3EI_x}$

$$=\frac{10\times\cos 30^\circ\times 60^3}{3\times 2\times 10^6\times 13\cdot 5}=0\cdot 0231\text{ cm}$$

Deflection along XX, $\delta_x=\dfrac{W\sin 30^\circ . L^3}{3EI_y}$

$$=\frac{10\times\sin 30^\circ\times 60^3}{3\times 2\times 10^6\times 54}=0\cdot 00333\text{ cm}$$

Horizontal deflection, $\vec{\delta}_H=\delta_y\sin 30^\circ-\delta_x\cos 30^\circ$

$$=0\cdot 0231\sin 30^\circ-0\cdot 00333\cos 30^\circ$$

$$=\mathbf{0\cdot 00867\ cm}$$

Vertical deflection, $\delta_v\downarrow=\delta_y\cos 30^\circ+\delta_x\sin 30^\circ$

$$=0\cdot 0231\cos 30^\circ+0\cdot 00333\sin 30^\circ$$

$$=\mathbf{0\cdot 0217\ cm.}$$

2. A 250 mm × 125 mm I beam having $I_x=3717\cdot 8$ cm⁴ and $I_y=193\cdot 4$ cm⁴ is used as a purlin on a roof truss. The angle of inclination of the main rafter of the roof is 30° to the horizontal and the rafters are placed 4 metres apart. If the vertical load on the purlin is 1,000 kg uniformly distributed, determine the maximum stress and the maximum vertical deflection of the purlin. (Purlins may be taken as simply supported at the ends.) $E=2\times 10^6$ kg/cm².

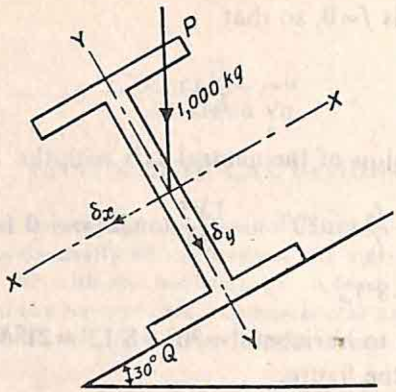

Fig. 52

$$\text{Maximum } M = \frac{1{,}000 \times 400}{8} = 50{,}000 \text{ kg cm}$$

$$f = \frac{M\cos 30^\circ}{I_x}.y + \frac{M\sin 30^\circ}{I_y}.x$$

Maximum compressive stress will occur at P and maximum tensile stress at Q and each is

$$= 50{,}000\left(\frac{12{\cdot}5\cos 30^\circ}{3717{\cdot}8} + \frac{6{\cdot}25\sin 30^\circ}{193{\cdot}4}\right)$$

$$= \mathbf{954\ kg/cm^2}$$

Deflection along YY,

$$\delta_y = \frac{5 \times 1{,}000\cos 30^\circ \times 400^3}{384 \times 2 \times 10^6 \times 3717{\cdot}8} = 0{\cdot}0971 \text{ cm}$$

Deflection along XX,

$$\delta_x = \frac{5 \times 1{,}000\sin 30^\circ \times 400^3}{384 \times 2 \times 10^6 \times 193{\cdot}4} = 1{\cdot}077 \text{ cm}$$

Vertical deflection, $\delta_V = \delta_y\cos 30^\circ + \delta_x\sin 30^\circ$

$$= \mathbf{0{\cdot}623\ cm.}$$

3. A roof is formed of rafters spaced 3 m centres and inclined at 30°. Purlins of channel section, 300 mm × 100 mm × 12 mm, are fixed to the rafters at intervals of 2 m. The wind load is 100 kg/m² normal to the plane of the rafters, and the dead load is 75 kg/m² of the same plane, but acting in a vertical direction. Find the maximum tensile and compressive stresses in the purlins. (Lond. Univ.)

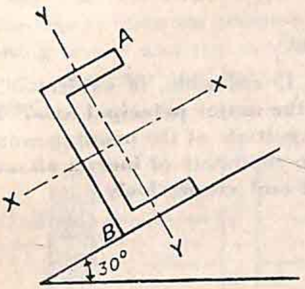

Fig. 53

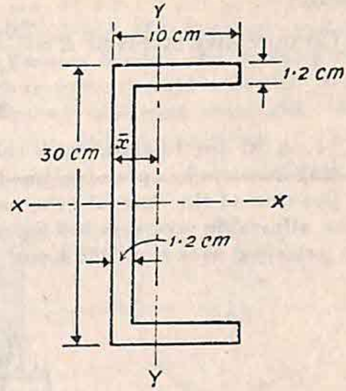

Fig. 54

Wind load along $YY = 100 \times 2 = 200$ kg/m

Dead load in the vertical direction $= 75 \times 2 = 150$ kg/m

Total load along $YY = 200 + 150\cos 30° = 330$ kg/m

Load along $XX = 150 \sin 30° = 75$ kg/m

The purlins are fixed at the rafters.

Moment in the plane YY,

$$M_x = \frac{wL^2}{12} = \frac{330 \times 9 \times 100}{12} = 24{,}750 \text{ kg cm},$$

causing bending about XX.

Moment in the plane XX,

$$M_y = \frac{75 \times 9 \times 100}{12} = 5{,}625 \text{ kg cm},$$

causing bending about YY.

Referring to Fig. 54,

$$(10 \times 1{\cdot}2 \times 2 + 27{\cdot}6 \times 1{\cdot}2)\bar{x} = 10 \times 1{\cdot}2 \times 5 \times 2 + 27{\cdot}6 \times 1{\cdot}2 \times 0{\cdot}6$$

$$\therefore \quad \bar{x} = 2{\cdot}45 \text{ cm}$$

$$I_x = \frac{10 \times 30^3}{12} - \frac{8{\cdot}8 \times 27{\cdot}6^3}{12} = 7{,}080 \text{ cm}^4$$

$$I_y = \tfrac{1}{3}(30 \times 2{\cdot}45^3 - 27{\cdot}6 \times 1{\cdot}25^3 + 1{\cdot}2 \times 7{\cdot}55^3 \times 2) = 473 \text{ cm}^4$$

Maximum tensile stress occurs at A and maximum compressive stress at B, Fig. 53.

For the point A, $x = 7{\cdot}55$ cm, $y = 15$ cm

Tensile stress at $A = \frac{24{,}750}{7{,}080} \times 15 + \frac{5{,}625}{473} \times 7{\cdot}55 = \mathbf{142{\cdot}3\ kg/cm^2}$

For the point B, $x = -2{\cdot}45$ cm, $y = -15$ cm

$$\text{Compressive stress at } B = \frac{24{,}750}{7{,}080} \times (-15) + \frac{5{,}625}{473} \times (-2{\cdot}45)$$

$$= -81{\cdot}5 \text{ kg/cm}^2.$$

4. A 30 cm I-beam, with the flanges 15 cm wide, is subjected to a bending moment in a plane inclined at 30° to the major principal axis. Find the position of the neutral axis, and the magnitude of the bending moment if the allowable stress is 800 kg/cm². The moments of inertia about the two principal axes are 7332·9 cm⁴ and 376·2 cm⁴ respectively.

(Lond. Univ.)

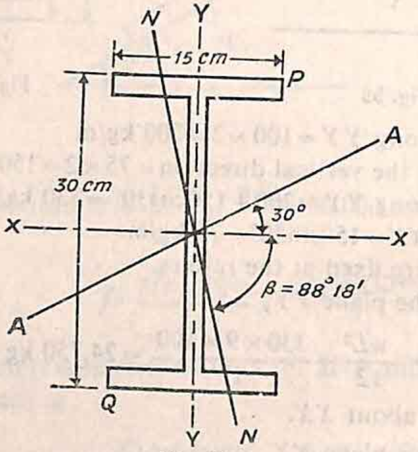

Fig. 55

$$f = \frac{M\sin 30°}{I_x} . y + \frac{M\cos 30°}{I_y} . x$$

At the neutral axis $f=0$, so that

$$y = -\frac{I_x}{I_y} \cot 30° . x$$

$$\therefore \quad \tan\beta = -\frac{I_x}{I_y} \cot 30° = -\frac{7332{\cdot}9}{376{\cdot}2} \cot 30° = -33{\cdot}8$$

$$\therefore \quad \beta = -88°18'$$

The neutral axis NN is shown in Fig. 55.

The maximum stress will occur at P and Q, so that

$$800 = M\left(\frac{15\sin 30°}{7332{\cdot}9} + \frac{7{\cdot}5\cos 30°}{376{\cdot}2}\right)$$

$$\therefore \quad M = 43{,}800 \text{ kg cm}.$$

5. A steel bar of rectangular section 10 cm×4 cm is supported in bearings and carries a load of 1,000 kg at mid-span. If as indicated in Fig. 56 the beam is rotated slowly, find the inclination θ when the bending stress in the bar reaches its greatest value. Determine the value of the greatest bending stress and the vertical deflection at mid-span when this stress occurs. Assume the bar is direction-free at the supports. $E=2\times10^6$ kg/cm². (Lond. Univ.)

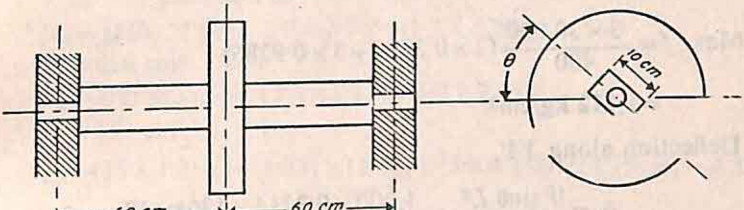

Fig. 56

Max. M at the centre of the span

$$=\frac{WL}{4}=\frac{1{,}000\times120}{4}=30{,}000 \text{ kg cm}$$

The components of M in the planes YY and XX are $M\sin\theta$ and $M\cos\theta$ respectively (Fig. 57)

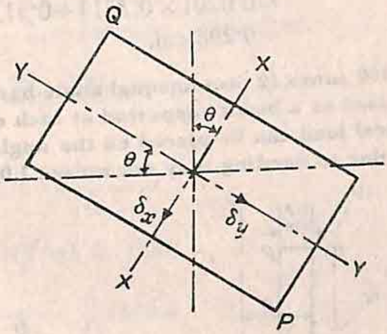

Fig. 57

The greatest tensile stress occurs at P and the greatest compressive stress at Q and each is given by

$$f=\frac{M\sin\theta}{Z_x}+\frac{M\cos\theta}{Z_y}=\frac{6M\sin\theta}{4\times100}+\frac{6M\cos\theta}{10\times16}$$

$$=\frac{3M}{400}(2\sin\theta+5\cos\theta)$$

For maximum stress, $\frac{df}{d\theta}=0$

or
$$2\cos\theta-5\sin\theta=0$$
or
$$\tan\theta=0{\cdot}4$$
$$\therefore\quad \theta=21^\circ\ 48'$$
$$\sin\theta=0{\cdot}3714,\ \cos\theta=0{\cdot}9285$$

$$\text{Max. } f=\frac{3\times 30{,}000}{400}(2\times 0{\cdot}3714+5\times 0{\cdot}9285)$$
$$=\mathbf{1{,}212\ kg/cm^2}$$

Deflection along YY,

$$\delta_y=\frac{W\sin\theta.L^3}{48EI_x}=\frac{1{,}000\times 0{\cdot}3714\times 120^3\times 12}{48\times 2\times 10^6\times 4\times 10^3}$$
$$=0{\cdot}0201\text{ cm}$$

Deflection along XX,

$$\delta_x=\frac{W\cos\theta.L^3}{48EI_y}=\frac{1{,}000\times 0{\cdot}9285\times 120^3\times 12}{48\times 2\times 10^6\times 10\times 4^3}$$
$$=0{\cdot}313\text{ cm}$$

Vertical deflection, $\delta_V=\delta_y\sin\theta+\delta_x\cos\theta$
$$=0{\cdot}0201\times 0{\cdot}3714+0{\cdot}313\times 0{\cdot}9285$$
$$=\mathbf{0{\cdot}298\ cm.}$$

6. An 150 mm × 100 mm × 12 mm unequal angle-bar is placed with the long leg vertical and used as a beam supported at each end, the span being 3 metres. What central load can be placed on the angle-bar in order that the maximum stress due to bending may not exceed 1,000 kg/cm² ?

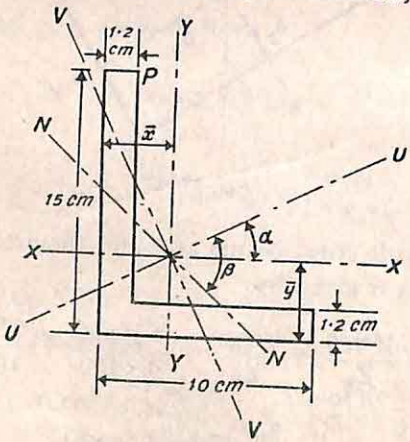

Fig. 58

$$A = 15\times 1{\cdot}2 + 8{\cdot}8\times 1{\cdot}2 = 28{\cdot}56 \text{ cm}^2$$

$$28{\cdot}56\bar{x} = 15\times 1{\cdot}2\times 0{\cdot}6 + 8{\cdot}8\times 1{\cdot}2\times 5{\cdot}6$$

$$\therefore \quad \bar{x} = 2{\cdot}45 \text{ cm}$$

$$28{\cdot}56\bar{y} = 15\times 1{\cdot}2\times 7{\cdot}5 + 8{\cdot}8\times 1{\cdot}2\times 0{\cdot}6$$

$$\therefore \quad \bar{y} = 4{\cdot}95 \text{ cm}$$

$$I_x = \tfrac{1}{3}(10\times 4{\cdot}95^3 - 8{\cdot}8\times 3{\cdot}75^3 + 1{\cdot}2\times 10{\cdot}05^3)$$
$$= 656 \text{ cm}^4$$

$$I_y = \tfrac{1}{3}(15\times 2{\cdot}45^3 - 13{\cdot}8\times 1{\cdot}25^3 + 1{\cdot}2\times 7{\cdot}55^3)$$
$$= 237 \text{ cm}^4$$

$$I_{xy} = (15\times 1{\cdot}2)\times(-1{\cdot}85)\times(2{\cdot}55) + (8{\cdot}8\times 1{\cdot}2)\times(3{\cdot}15)\times(-4{\cdot}35)$$
$$= -230 \text{ cm}^4$$

$$\tan 2\alpha = -\frac{2I_{xy}}{I_x - I_y} = \frac{2\times 230}{656-237} = 1{\cdot}098$$

$$\therefore \quad 2\alpha = 47^\circ 42'$$
$$\alpha = 23^\circ\ 51'$$

$$I_u + I_v = I_x + I_y = 893 \qquad \cdot\cdot \quad (1)$$

$$I_u - I_v = (I_x - I_y)\sec 2\alpha = 419\times 1{\cdot}4859$$
$$= 623 \qquad \cdot\cdot \quad (2)$$

Solving eqs. 1 and 2

$$I_u = 758 \text{ cm}^4,\ I_v = 135 \text{ cm}^4$$

$$\text{Max. } M = \frac{WL}{4} = \frac{W\times 300}{4} = 75W \text{ kg cm}$$

$$f = \frac{M\cos\alpha}{I_u}.\ v + \frac{M\sin\alpha}{I_v}.u$$

At the neutral axis $f=0$, so that

$$v = -\frac{I_u}{I_v}\tan\alpha.u$$

If β is the inclination of the neutral axis with UU, then

$$\tan\beta = -\frac{I_u}{I_v}\tan\alpha = -\frac{758}{135}\times 0{\cdot}4421$$
$$= -2{\cdot}48$$
$$\therefore \quad \beta = -68^\circ$$

By inspection it is found that the point P is farthest from the neutral axis. The coordinates of P referred to UU and VV can be found out by drawing or calculation.

By calculation

$$u = 10{\cdot}05\sin\alpha - 1{\cdot}25\cos\alpha = 2{\cdot}92 \text{ cm}$$
$$v = 10{\cdot}05\cos\alpha + 1{\cdot}25\sin\alpha = 9{\cdot}70 \text{ cm}$$

Stress at P, $$1{,}000 = M\left(\frac{9{\cdot}70\cos\alpha}{758} + \frac{2{\cdot}92\sin\alpha}{135}\right)$$

$\therefore$ $$M = 48{,}900 \text{ kg cm}$$

$$W = \frac{48{,}900}{75} = \mathbf{652\ kg.}$$

7. Fig. 59 shows an unequal angle section, for which $I_x = 80{\cdot}8$ cm⁴ and $I_y = 38{\cdot}8$ cm⁴. Find the moments of inertia about the principal axes UU and VV, given that the angle between the axes UU and XX is 28° 30′.

If the angle, with the 80 mm leg vertical, is used as a beam, freely supported on a span of 2 m carrying a vertical load of 200 kg at the centre, find (a) the maximum bending stress at the point A; (b) the direction and magnitude of the maximum deflection.

Neglect the weight of the beam and take $E = 2 \times 10^6$ kg/cm².

(Lond. Univ.)

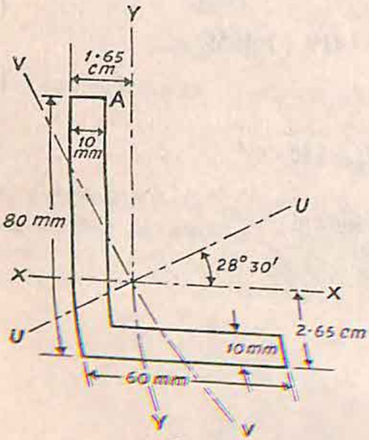

Fig. 59

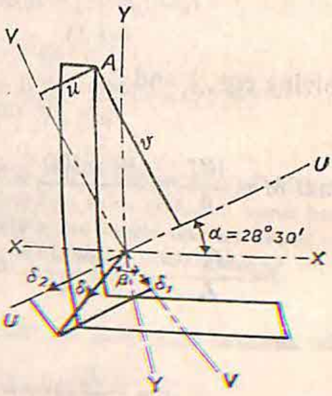

Fig. 60

$$I_u = \tfrac{1}{2}(I_x + I_y) + \tfrac{1}{2}(I_x - I_y)\sec 2\alpha$$
$$= \tfrac{1}{2} \times 119{\cdot}6 + \tfrac{1}{2} \times 42 \times 1{\cdot}8361$$
$$= \mathbf{98{\cdot}4\ cm^4}$$
$$I_v = \tfrac{1}{2}(I_x + I_y) - \tfrac{1}{2}(I_x - I_y)\sec 2\alpha$$
$$= \tfrac{1}{2} \times 119{\cdot}6 - \tfrac{1}{2} \times 42 \times 1{\cdot}8361$$
$$= \mathbf{21{\cdot}2\ cm^4}$$

(a) Maximum $M = \dfrac{200 \times 200}{4} = 10{,}000$ kg cm

The coordinates of A, referred to the axes UU and VV, can be found out by drawing or calculation.

By calculation : Fig. 60

$$x = -0{\cdot}65 \text{ cm},\ y = +5{\cdot}35 \text{ cm}$$

$$u = 5{\cdot}35 \sin 28°30' - 0{\cdot}65 \cos 28°30' = 1{\cdot}98 \text{ cm}$$

$$v = 5{\cdot}35 \cos 28°30' + 0{\cdot}65 \sin 28°30' = 5{\cdot}01 \text{ cm}$$

$$f = M\left(\frac{v\cos\alpha}{I_u} + \frac{u\sin\alpha}{I_v}\right)$$

$$= 10{,}000\left(\frac{5{\cdot}01\cos 28°30'}{98{\cdot}4} + \frac{1{\cdot}98\sin 28°30'}{21{\cdot}2}\right)$$

$$= \mathbf{893\ kg/cm^2}.$$

(b) Deflection along VV, $\delta_1 = \dfrac{W\cos\alpha . L^3}{48EI_u}$

Deflection along UU, $\delta_2 = \dfrac{W\sin\alpha . L^3}{48EI_v}$

Resultant deflection, $\delta = \sqrt{\delta_1^2 + \delta_2^2}$

$$= \frac{WL^3}{48E}\sqrt{\left(\frac{\cos\alpha}{I_u}\right)^2 + \left(\frac{\sin\alpha}{I_v}\right)^2}$$

$$= \frac{200 \times 200^3}{48 \times 2 \times 10^6}\sqrt{\left(\frac{\cos 28°30'}{98{\cdot}4}\right)^2 + \left(\frac{\sin 28°30'}{21{\cdot}2}\right)^2}$$

$$= \mathbf{0{\cdot}403\ cm}$$

$$\tan\beta = \frac{\delta_2}{\delta_1} = \frac{I_u}{I_v}\tan\alpha = \frac{98{\cdot}4}{21{\cdot}2}\tan 28°30'$$

$$= 2{\cdot}52$$

$$\therefore \quad \beta = 68°21'$$

$\therefore$ Angle to vertical $= 68°21' - 28°30' = \mathbf{39°51'}$.

8. A beam of angle section 150 mm × 115 mm × 12 mm, shown in Fig. 61, is freely supported on a span of 2 m with the 150 mm leg vertically downwards. A vertical load of 1,200 kg is applied on the 115 mm leg at the centre of the span. Find the maximum tensile and compressive stresses in the beam.

The properties of the section are: $I_x = 676{\cdot}5$ cm^4, $I_y = 345{\cdot}3$ cm^4, $\tan\alpha = 0{\cdot}58$.

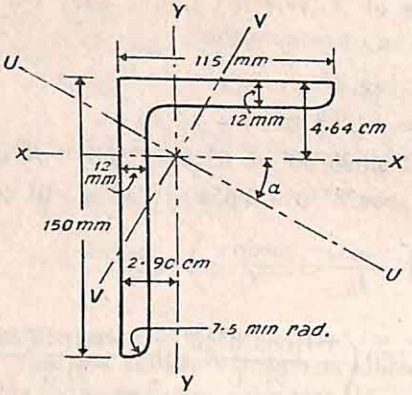

Fig. 61

$$\tan\alpha = 0{\cdot}58, \quad \therefore \quad \alpha = 30^\circ\ 6'$$

$$I_u + I_v = I_x + I_y = 1021{\cdot}8 \qquad \ldots \quad (1)$$

$$I_u - I_v = (I_x - I_y)\sec 2\alpha = 331{\cdot}2 \times \sec 60^\circ\ 12'$$

$$= 666{\cdot}4 \qquad \ldots \quad (2)$$

Solving eqs. 1 and 2,

$$I_u = 844{\cdot}1 \text{ cm}^4, \quad I_v = 177{\cdot}7 \text{ cm}^4$$

$$M = \frac{1{,}200 \times 200}{4} = 60{,}000 \text{ kg cm}$$

$$f = -\frac{M\cos\alpha}{I_u}.v + \frac{M\sin\alpha}{I_v}.u$$

At the neutral axis $f=0$, so that

$$v = \frac{I_u}{I_v}\tan\alpha.u$$

If β is the inclination of the neutral axis NN with UU, then

$$\tan\beta = \frac{I_u}{I_v}\tan\alpha = \frac{844{\cdot}1}{177{\cdot}7} \times 0{\cdot}58$$

$$= 2{\cdot}755$$

$$\therefore \quad \beta = 70^\circ\ 3'$$

The neutral axis NN is drawn as shown in Fig. 62. The points on the section farthest from the neutral axis are P and Q.

By drawing the co-ordinates of P are $+3{\cdot}4$ cm, $-9{\cdot}8$ cm and those of Q are $-4{\cdot}8$ cm. $+2{\cdot}6$ cm.

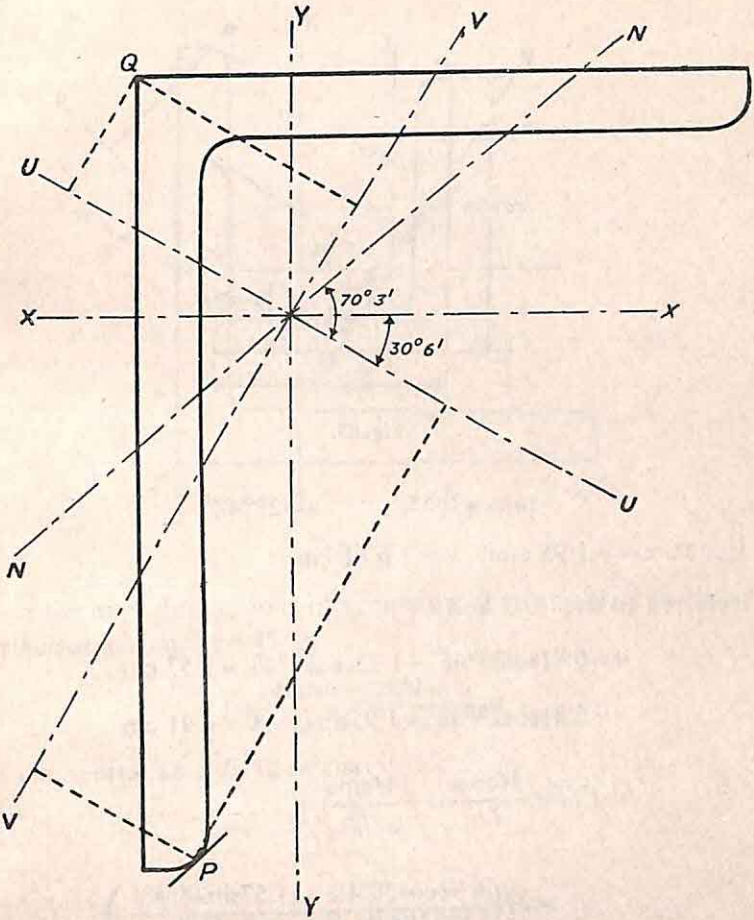

Fig. 62

Maximum tension at P

$$=60{,}000\left(\frac{9{\cdot}8\cos 30°6'}{844{\cdot}1}+\frac{3{\cdot}4\sin 30°6'}{177{\cdot}7}\right)$$

$$=\mathbf{1{,}178\ kg/cm^2}$$

Maximum compression at Q

$$=60{,}000\left(-\frac{2{\cdot}6\cos 30°6'}{844{\cdot}1}-\frac{4{\cdot}8\sin 30°6'}{177{\cdot}7}\right)$$

$$=\mathbf{-974\ kg/cm^2}.$$

9. Fig. 63 shows the section of an unequal angle iron. It is subjected to a bending moment M, the plane of which has GY as trace. Calculate the longitudinal stress at the point P. The principal axes are GU and GV; the moment of inertia about GV is 41·2 cm⁴; that about GU is 196·1 cm⁴ and $\tan\alpha=0{\cdot}55$.

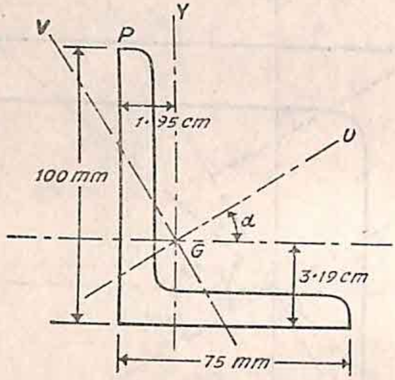

Fig. 63

$$\tan\alpha = 0{\cdot}55, \quad \therefore \quad \alpha = 28°48'$$

For P, $x = -1{\cdot}95$ cm, $y = +6{\cdot}81$ cm

Referred to axes GU and GV,

$$u = 6{\cdot}81\sin 28°48' - 1{\cdot}95\cos 28°48' = 1{\cdot}57 \text{ cm}$$

$$v = 6{\cdot}81\cos 28°48' + 1{\cdot}95\sin 28°48' = 6{\cdot}91 \text{ cm}$$

$$\text{At } P,\ f = \frac{M\cos\alpha}{I_u}.v + \frac{M\sin\alpha}{I_v}.u$$

$$= M\left(\frac{6{\cdot}91\cos 28°48'}{196{\cdot}1} + \frac{1{\cdot}57\sin 28°48'}{41{\cdot}2}\right)$$

$$= 0{\cdot}0492\ \text{M}.$$

10. An 80 mm×80 mm×10 mm angle is used as a beam simply supported at each end over a span of 2 m, with one leg of the section horizontal and the other vertically upwards. It is loaded at the centre of the span with a vertical load which may be assumed to pass through the centroid of the section. The principal second moments of area for the section are 139·5 cm^4 and 36·0 cm^4. The distance of the centroid from the outside edge is 2·34 cm and the toe has a radius of 4·5 mm. Find the position of the neutral axis and calculate the safe load if the maximum stress is not to exceed 1,200 kg/cm^2.

(Lond. Univ.)

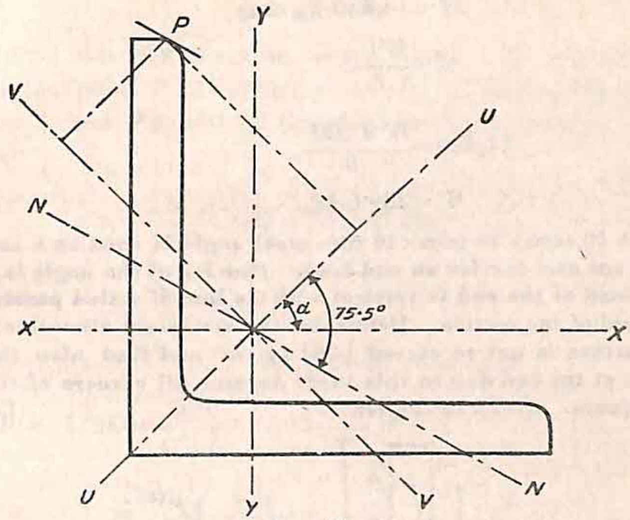

Fig. 64

Since the angle has an axis of symmetry, that axis (UU in Fig. 64) is a principal axis and $\alpha = 45°$.

$$f = \frac{M\cos\alpha}{I_u}.v + \frac{M\sin\alpha}{I_v}.u$$

At the neutral axis $f=0$, so that

$$v = -\frac{I_u}{I_v}\tan\alpha.u$$

$$= -\frac{I_u}{I_v}.u \quad \text{since } \tan\alpha = 1$$

$$\therefore \quad \tan\beta = -\frac{I_u}{I_v} = -\frac{139{\cdot}5}{36} = -3{\cdot}88$$

$$\therefore \quad \beta = 75{\cdot}5°$$

The neutral axis NN is drawn as shown in figure and by inspection it is found that the point P is the farthest point in the section from NN.

For P, $u = 2{\cdot}9$ cm, $v = 5{\cdot}1$ cm.

$$f = M\left(\frac{v\cos\alpha}{I_u} + \frac{u\sin\alpha}{I_v}\right)$$

$$1{,}200 = \frac{M}{\sqrt{2}}\left(\frac{5{\cdot}1}{139{\cdot}5} + \frac{2{\cdot}9}{36}\right)$$

$$\therefore \quad M = 14{,}480 \text{ kg cm}$$

But
$$M = \frac{WL}{4}$$

or
$$14{,}480 = \frac{W \times 200}{4}$$

$$\therefore \quad W = 289{\cdot}6 \text{ kg.}$$

11. A 90 mm × 90 mm × 10 mm steel angle is used as a cantilever of length 80 cm and carries an end load. One leg of the angle is horizontal and the load at the end is vertical with its line of action passing through the centroid of the section. Determine the maximum allowable load if the bending stress is not to exceed 1,200 kg/cm² and find also the vertical deflection at the end due to this load. Assume all corners of the angle to be left square. $E = 2 \times 10^6$ kg/cm².

(Lond. Univ.)

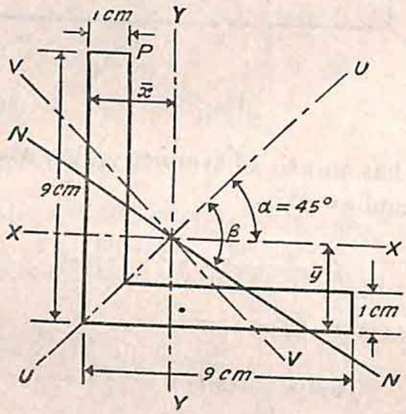

Fig. 65

$$A = 9 \times 1 + 8 \times 1 = 17 \text{ cm}^2$$

$$17\bar{x} = 9 \times 1 \times 4{\cdot}5 + 8 \times 1 \times 0{\cdot}5$$

$$\therefore \quad \bar{x} = 2{\cdot}62 \text{ cm} = \bar{y}$$

$$I_x = I_y = \tfrac{1}{3}(9 \times 2{\cdot}62^3 - 8 \times 1{\cdot}62^3 + 1 \times 6{\cdot}38^3)$$
$$= 129{\cdot}2 \text{ cm}^4$$

Since the angle has an axis of symmetry, that axis (*UU* in Fig. 65) is a principal axis and $\alpha = 45°$.

$$I_u = \tfrac{1}{12}(9^4 - 8^4) = 205{\cdot}4 \text{ cm}^4$$

Also $I_u + I_v = I_x + I_y = 258{\cdot}4$

$$\therefore \quad I_v = 53 \text{ cm}^4$$

Maximum M at the support $= 80W$

$$\tan\beta = -\frac{I_u}{I_v}\tan\alpha = -\frac{205{\cdot}4}{53} = -3{\cdot}875$$

$$\therefore \quad \beta = -75° \ 30'.$$

The neutral axis NN is shown in the figure. By inspection it is found that the point P is farthest from NN. The co-ordinates of P referred to UU and VV can be found either from drawing or from calculation.

$$u = \frac{8}{\sqrt{2}} - 1{\cdot}62\sqrt{2} = 3{\cdot}37 \text{ cm}$$

$$v = \frac{8}{\sqrt{2}} = 5{\cdot}66 \text{ cm}$$

Stress at P, $1{,}200 = \dfrac{80W}{\sqrt{2}.I_u}.v + \dfrac{80W}{\sqrt{2}.I_v}.u$

$$= \frac{80W}{\sqrt{2}}\left(\frac{5{\cdot}66}{205{\cdot}4} + \frac{3{\cdot}37}{53}\right)$$

$$\therefore \quad W = 233 \text{ kg}$$

Deflection along VV,

$$\delta_1 = \frac{WL^3}{\sqrt{2} \times 3EI_u} = \frac{233 \times 80^3}{\sqrt{2} \times 3 \times 2 \times 10^6 \times 205{\cdot}4}$$

$$= 0{\cdot}0685 \text{ cm}$$

Deflection along UU,

$$\delta_2 = \frac{WL^3}{\sqrt{2} \times 3EI_v} = \frac{233 \times 80^3}{\sqrt{2} \times 3 \times 2 \times 10^6 \times 53}$$

$$= 0{\cdot}265 \text{ cm}$$

Vertical deflection,

$$\delta_v = \frac{\delta_1}{\sqrt{2}} + \frac{\delta_2}{\sqrt{2}} = 0{\cdot}236 \text{ cm.}$$

12) A cantilever consists of an 80 mm × 80 mm × 12 mm angle with the top face AB horizontal as shown in Fig. 66. It carries a load of 200 kg at 1 m from the fixed end, the line of action of the load passing through the centroid of the section and inclined at 30° to the vertical. Determine the stresses at the corners A, B and C at the fixed end and also the position of the neutral axis.

$I_x = I_y = 101{\cdot}9$ **cm⁴.**

$I_u = 161{\cdot}4$ **cm⁴,** $I_v = 42{\cdot}4$ **cm⁴.**

(Lond. Univ.)

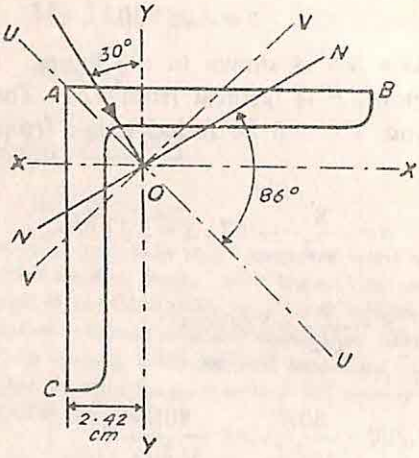

Fig. 66

At the fixed end, $M=200\times 100=20{,}000$ kg cm

$$f=\frac{M\cos 75^\circ}{I_u}.v-\frac{M\sin 75^\circ}{I_v}.u$$

$$=20{,}000\left(\frac{v\cos 75^\circ}{161{\cdot}4}-\frac{u\sin 75^\circ}{42{\cdot}4}\right)$$

$$=32{\cdot}1v-456u$$

At A, $u=-2{\cdot}42\sqrt{2}=-3{\cdot}42$ cm

$v=0$

$\therefore$ $f=456\times 3{\cdot}42=$ **1,560 kg/cm² (tensile)**

At B, $u=\dfrac{8}{\sqrt{2}}-2{\cdot}42\sqrt{2}=2{\cdot}23$ cm

$$v=\frac{8}{\sqrt{2}}=5{\cdot}66 \text{ cm}$$

$\therefore$ $f=32{\cdot}1\times 5{\cdot}66-456\times 2{\cdot}23$

$=$ **-835 kg/cm² (compressive)**

At C, $u=2{\cdot}23$ cm, $v=-5{\cdot}66$ cm

$\therefore$ $f=-32{\cdot}1\times 5{\cdot}66-456\times 2{\cdot}23$

$=$ **$-1{,}199$ kg/cm² (compressive)**

At the neutral axis, $f=0$, so that

$$v=\frac{456}{32{\cdot}1}u=14{\cdot}21u$$

If β is the inclination of NN with UU,

$$\tan\beta=14{\cdot}21 \quad \therefore \quad \beta=+86^\circ$$

The neutral axis NN is shown in the figure. The stress will be tensile above the neutral axis and compressive below.

13. A 100 mm×100 mm angle as shown in Fig. 67 is used as a freely supported beam with one leg vertical. $I_x = I_y = 207$ cm⁴. When a bending moment is applied in the vertical plane YY the mid-section of the beam deflects in the direction AA. Calculate the second moments of area of the section about its principal axes. Find also the bending stress at the upper corner of the section if the bending moment is 400 kg m. (Lond. Univ.)

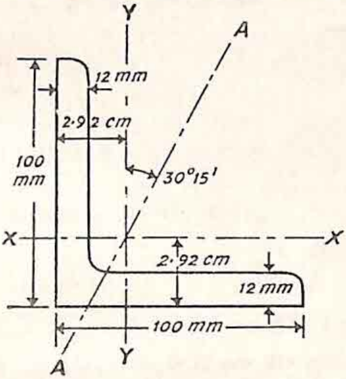

Fig. 67

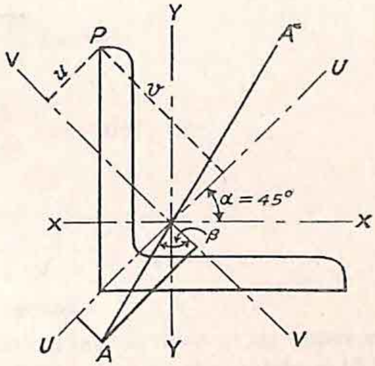

Fig. 68

$$I_u + I_v = I_x + I_y = 414 \text{ cm}^4 \quad \ldots \quad (1)$$

Deflection along VV, $\delta_1 = \dfrac{W\cos\alpha L^3}{48EI_u}$

Deflection along UU, $\delta_2 = \dfrac{W\sin\alpha L^3}{48EI_v}$

$$\tan\beta = \frac{\delta_2}{\delta_1} = \frac{I_u}{I_v}\tan\alpha$$

$$= \frac{I_u}{I_v}, \text{ since } \tan\alpha = 1$$

$$\beta = 45° + 30°15' = 75°15'$$

$$\therefore \quad \frac{I_u}{I_v} = \tan 75°15' = 3{\cdot}80 \quad \ldots \quad (2)$$

From eqs. 1 and 2

$$I_u = 327{\cdot}7 \text{ cm}^4, \ I_v = 86{\cdot}3 \text{ cm}^4$$

For P, $u = \dfrac{10}{\sqrt{2}} - 2{\cdot}92\sqrt{2} = 2{\cdot}94$ cm

$$v = \frac{10}{\sqrt{2}} = 7{\cdot}07 \text{ cm}$$

$$f=M\left(\frac{v\cos\alpha}{I_u}+\frac{u\sin\alpha}{I_v}\right)$$

$$=\frac{400\times100}{\sqrt{2}}\left(\frac{7\cdot07}{327\cdot7}+\frac{2\cdot94}{86\cdot3}\right)$$

$$=\mathbf{1{,}575\ kg/cm^2.}$$

CHAPTER VIII

VIBRATIONS AND CRITICAL SPEEDS

1. A 30 kg weight is suspended from a rod 2·5 cm diameter and 2 m long. Neglecting the mass of the rod, find the frequency of free longitudinal vibration. $E = 2 \times 10^6$ kg/cm².

Stress $\quad f = \dfrac{30 \times 4}{\pi \times 2{\cdot}5^2} = \dfrac{96}{5\pi}$ kg/cm²

Elongation $\quad \delta = \dfrac{96 \times 200}{5\pi \times 2 \times 10^6} = 0{\cdot}000611$ cm

Frequency of vibration,

$$N = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}000611}}$$

$$= 201{\cdot}7 \text{ per second.}$$

2. A steel wire 0·32 cm diameter, 60 cm long is fixed at its upper end and carries at its lower end a horizontal disc of which the weight is 15 kg, and radius of gyration 9 cm. It is found by observation that the disc makes 20 complete oscillations in 37·8 seconds. Find the modulus of rigidity of the wire.

$$I = \frac{W}{g}k^2 = \frac{15 \times 9^2}{981} = \frac{135}{109} \text{ kg cm sec}^2$$

$$N = \frac{1}{2\pi}\sqrt{\frac{CJ}{IL}}$$

or

$$\frac{20}{37{\cdot}8} = \frac{1}{2\pi}\sqrt{\frac{C \times \pi(0{\cdot}32)^4 \times 109}{32 \times 135 \times 60}}$$

$$\therefore \quad C = 0{\cdot}798 \times 10^6 \text{ kg/cm}^2.$$

3. A steel wire 1 metre long and 0·2 cm diameter is fixed at one end and carries at the other a short cast-iron cylinder 20 cm diameter, with its axis, which is 2 cm long, in line with the axis of the wire. Find the frequency of natural torsional oscillations of the cylinder, the weight of cast-iron being 7·2 g/cm³, and C for steel being $0{\cdot}8 \times 10^6$ kg/cm².

Weight of cylinder,

$$W = \frac{\pi}{4} \times 20^2 \times 2 \times \frac{7{\cdot}2}{1{,}000} = 4{\cdot}52 \text{ kg}$$

$$I = \frac{W}{g}k^2 = \frac{W}{g} \times \frac{d^2}{8}$$

$$=\frac{4{\cdot}52\times 20^2}{981\times 8}=\frac{226}{981}\text{ kg cm sec}^2$$

$$N=\frac{60}{2\pi}\sqrt{\frac{CJ}{IL}}=\frac{30}{\pi}\sqrt{\frac{0{\cdot}8\times 10^6\times\pi\times(0{\cdot}2)^4\times 981}{226\times 100\times 32}}$$

$$=22{\cdot}3\text{ per minute.}$$

4. A steel shaft 8 cm in diameter carries two flywheels weighing 1,200 kg and 300 kg respectively. Length of the shaft between flywheels is 1·5 m. Determine the natural frequency of longitudinal vibration. $E=2\times 10^6$ kg/cm².

(Engineering Services, 1964)

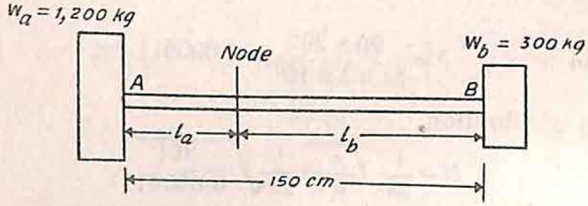

Fig. 69

Let the node divide the shaft into two parts l_a and l_b. Then the frequency of vibration of the system to the left of node is given by

$$N_a=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}=\frac{1}{2\pi}\sqrt{\frac{gEA}{W_a.l_a}}$$

and the frequency of vibration of the system to the right of node is given by

$$N_b=\frac{1}{2\pi}\sqrt{\frac{gEA}{W_b.l_b}}$$

But these two frequencies must be equal.

$$\therefore\quad W_a.l_a=W_b.l_b$$

or

$$1{,}200\times l_a=300(150-l_a)$$

$$\therefore\quad l_a=30\text{ cm}$$

Hence the frequency of vibration

$$=N_a=\frac{1}{2\pi}\sqrt{\frac{981\times 2\times 10^6\times\pi\times 64}{1{,}200\times 30\times 4}}$$

$$=263\text{ per sec.}$$

5. A shaft 60 cm long has a diameter of 4 cm for the first 20 cm and a diameter of 6 cm for the remaining 40 cm. If one end of the shaft is fixed and the other end carries a disc of weight 600 kg and radius of gyration 40 cm, what is the frequency of the free torsional oscillations? $C=0{\cdot}8\times 10^6$ kg/cm².

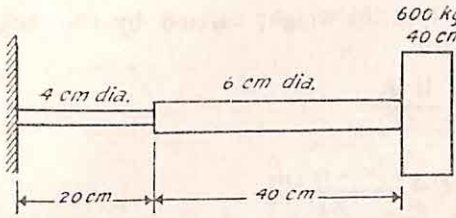

Fig. 70

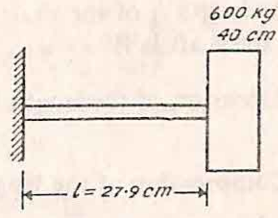

Fig. 71

Equivalent length of shaft of 4 cm diameter,

$$l=l_1+l_2\left(\frac{d_1}{d_2}\right)^4=20+40\left(\frac{4}{6}\right)^4$$

$$=27{\cdot}9 \text{ cm}$$

The system in Fig. 71 is then torsionally equivalent to the system in Fig. 70.

$$N=\frac{1}{2\pi}\sqrt{\frac{CJ}{Il}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{0{\cdot}8\times10^6\times\pi\times4^4\times981}{600\times40^2\times27{\cdot}9\times32}}$$

$$=4{\cdot}32 \text{ per second}$$

6. A flywheel is mounted on a vertical shaft as shown in Fig. 72, the ends of the shaft being fixed. The shaft is 5 cm diameter, the length l_1 is 90 cm and the length l_2 is 60 cm. The flywheel weighs 500 kg and its radius of gyration is 50 cm. Find the natural frequencies of the longitudinal, the transverse and the torsional vibrations of the system. $E=2\times10^6$ kg/cm², $C=0{\cdot}8\times10^6$ kg/cm².

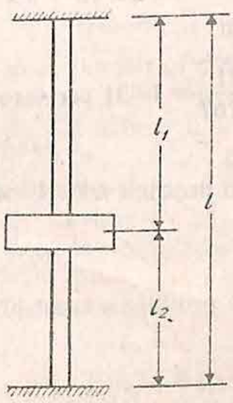

Fig. 72

(a) Longitudinal vibration :

Let W'_1 be the part of the weight W of the flywheel which is carried

by the length l_1 of the shaft, so that the weight carried by the length l_2 of the shaft is $W-W_1$

Extension of the length $l_1=\dfrac{W_1 l_1}{AE}$

Compression of the length $l_2=\dfrac{(W-W_1)l_2}{AE}$

But the extension of the length l_1 of the shaft will be equal to the compression of the length l_2 of the shaft, so that

$$W_1 l_1=(W-W_1)l_2$$

or

$$W_1\times 90=(500-W_1)\times 60$$

$$\therefore\quad W_1=200 \text{ kg}$$

Extension of the shaft l_1,

$$\delta=\frac{200\times 90\times 4}{\pi\times 5^2\times 2\times 10^6}=0{\cdot}000458 \text{ cm}$$

Frequency of vibrations,

$$N=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}=\frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}000458}}$$

$$=\mathbf{233 \text{ per second.}}$$

(b) Transverse Vibration :

The static deflection under the load, for a horizontal shaft fixed at the ends and loaded at a point which divides the shaft into the two parts l_1 and l_2 is given by

$$\delta=\frac{Wl_1^3 l_2^3}{3EIl^3}=\frac{500\times 90^3\times 60^3\times 64}{3\times 2\times 10^6\times\pi\times 5^4\times 150^3}$$

$$=0{\cdot}1267 \text{ cm}$$

$$\therefore\quad N=\frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}1267}}=\mathbf{14{\cdot}01 \text{ per second}}.$$

(c) Torsional Vibration :

The torque required to produce a twist of 1 radian in the length l_1 of the shaft $=\dfrac{CJ}{l_1}$

and the torque required to produce a twist of 1 radian in the length l_2 $=\dfrac{CJ}{l_2}$

$\therefore$ Total torque required at the flywheel to produce a twist of 1 radian $=K=CJ\left(\dfrac{1}{l_1}+\dfrac{1}{l_2}\right)$

$$=0{\cdot}8\times10^6\times\frac{\pi}{32}\times5^4\left(\frac{1}{90}+\frac{1}{60}\right)$$

$$=\frac{625\pi\times10^5}{144}\text{ kg cm}$$

$$N=\frac{1}{2\pi}\sqrt{\frac{K}{I}}=\frac{1}{2\pi}\sqrt{\frac{625\pi\times10^5\times981}{144\times500\times50^2}}$$

$$=\mathbf{5{\cdot}21\ per\ second.}$$

7. A circular rod AB, 4 cm diameter and 2 m long, has circular discs rigidly fixed to its two ends. That at A has a mass of 20 kg and is of 25 cm diameter, and that at B has a mass of 75 kg and is of 40 cm diameter. The rod is held in a horizontal position by a clamp at a point C between A and B, the position of C being such that the frequencies of the transverse vibrations of the parts of the system on either side of C are equal. Determine the distance of C from A and the frequency of the vibrations.

Find also the ratio of the frequencies of the torsional vibrations. Young's modulus $E=2\times10^6$ kg/cm². Neglect the inertia of the rod.

(Lond. Univ.)

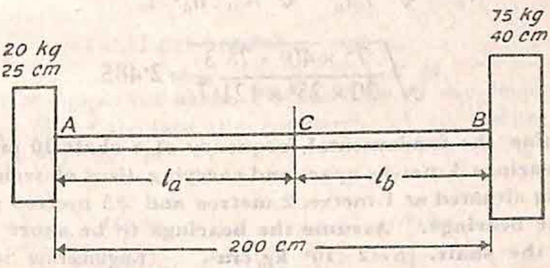

Fig. 73

Frequency of vibration to the left of C,

$$N_a=\frac{1}{2\pi}\sqrt{\frac{g}{\delta_a}}\text{ where }\delta_a=\frac{W_a l_a^3}{3EI}$$

Frequency of vibration to the right of C,

$$N_b=\frac{1}{2\pi}\sqrt{\frac{g}{\delta_b}}\text{ where }\delta_b=\frac{W_b l_b^3}{3EI}$$

But the two frequencies are equal.

$$\therefore\quad \delta_a=\delta_b$$

or
$$W_a l_a^3=W_b l_b^3$$

or
$$\frac{l_a}{l_b}=\sqrt[3]{\frac{W_b}{W_a}}=\sqrt[3]{\frac{75}{20}}=1{\cdot}554$$

Also
$$l_a+l_b=200\text{ cm}$$

Solving $l_a = \mathbf{121{\cdot}7}$ **cm**

$$\delta_a = \frac{20 \times 121{\cdot}7^3 \times 64}{3 \times 2 \times 10^6 \times \pi \times 4^4} = 0{\cdot}478 \text{ cm}$$

Frequency of vibrations,

$$= N_a = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_a}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}478}}$$

$$= \mathbf{7{\cdot}21 \text{ per second}}$$

Frequency of torsional vibration to the left of C,

$$N_a = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_a l_a}} \text{ where } I_a = \frac{W_a}{g} \cdot \frac{d_a^2}{8}$$

Frequency of torsional vibration to the right of C,

$$N_b = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_b l_b}} \text{ where } I_b = \frac{W_b}{g} \cdot \frac{d_b^2}{8}$$

$$\therefore \quad \text{Ratio } \frac{N_a}{N_b} = \sqrt{\frac{I_b l_b}{I_a l_a}} = \sqrt{\frac{W_b \cdot d_b^2 \cdot l_b}{W_a \cdot d_a^2 \cdot l_a}}$$

$$= \sqrt{\frac{75 \times 40^2 \times 78{\cdot}3}{20 \times 25^2 \times 121{\cdot}7}} = 2{\cdot}485.$$

8. Determine the fundamental frequency of a shaft 10 cm diameter, supported in bearings 3 metres apart and carrying discs of weights 100 kg, 150 kg and 75 kg situated at 1 metre, 2 metres and 2·5 metres respectively from one of the bearings. Assume the bearings to be short and neglect the inertia of the shaft. $E = 2 \times 10^6$ **kg/cm²**. (Engineering Services, 1967)

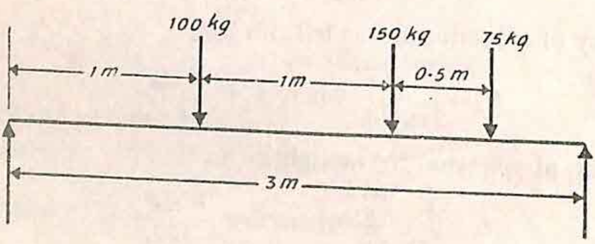

Fig. 74

Apply Dunkerley's formula.

For 100 kg load alone,

$$\delta_1 = \frac{Wa^2b^2}{3EIL} = \frac{100 \times 100^2 \times 200^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 10^4 \times 300}$$

$$= 0{\cdot}0453 \text{ cm}$$

For 150 kg load alone,

$$\delta_2 = \frac{150 \times 200^2 \times 100^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 10^4 \times 300} = 0{\cdot}0679 \text{ cm}$$

For 75 kg load alone,

$$\delta_3 = \frac{75 \times 250^2 \times 50^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 10^4 \times 300} = 0{\cdot}0133 \text{ cm}$$

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2}$$

$$= \frac{4\pi^2}{g}\delta_1 + \frac{4\pi^2}{g}\delta_2 + \frac{4\pi^2}{g}\delta_3 = \frac{4\pi^2}{g}(\delta_1 + \delta_2 + \delta_3)$$

$$= \frac{4\pi^2}{g}\Sigma\delta$$

$$\therefore \quad N = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}1265}}$$

$$= \mathbf{14{\cdot}02 \text{ per second.}}$$

9. A simply supported beam, 2·5 cm wide × 5 cm deep, is loaded as shown in Fig. 75. Calculate the frequency of its natural transverse vibration. $E = 2 \times 10^6$ kg/cm². (Engineering Services, 1967)

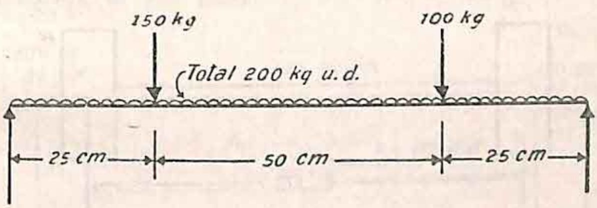

Fig. 75

Apply Dunkerley's formula.

For 150 kg load alone,

$$\delta_1 = \frac{150 \times 25^2 \times 75^2 \times 12}{3 \times 2 \times 10^6 \times 2{\cdot}5 \times 5^3 \times 100} = 0{\cdot}0338 \text{ cm}$$

For 100 kg load alone,

$$\delta_2 = \frac{100 \times 75^2 \times 25^2 \times 12}{3 \times 2 \times 10^6 \times 2{\cdot}5 \times 5^3 \times 100} = 0{\cdot}0225 \text{ cm}$$

Due to concentrated loads only the frequency of vibration is given by

$$N_o = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0.0563}}$$

$$= 21 \text{ per second.}$$

For the uniformly distributed load alone,

$$N_s = \frac{\pi}{2L^2}\sqrt{\frac{gEI}{w}}$$

$$w = \tfrac{200}{100} = 2 \text{ kg/cm}$$

$$\therefore \quad N_s = \frac{\pi}{2\times 100^2}\sqrt{\frac{981\times 2\times 10^6\times 2.5\times 5^3}{2\times 12}}$$

$$= 25.1 \text{ per second.}$$

Hence

$$\frac{1}{N^2} = \frac{1}{N_0^2} + \frac{1}{N_s^2}$$

$$= \frac{1}{21^2} + \frac{1}{25.1^2}$$

$$\therefore \quad N = 16.1 \text{ per second.}$$

10. Two flywheels weighing 1,400 kg and 1,200 kg, with radii of gyration 75 cm and 67·5 cm respectively, are keyed 90 cm apart on a steel shaft of diameter 10 cm. Neglecting the effect of the mass of the shaft, calculate the frequency of free torsional vibrations and the position of the node. $C = 0.8\times 10^6$ kg/cm². (Engineering Services, 1954)

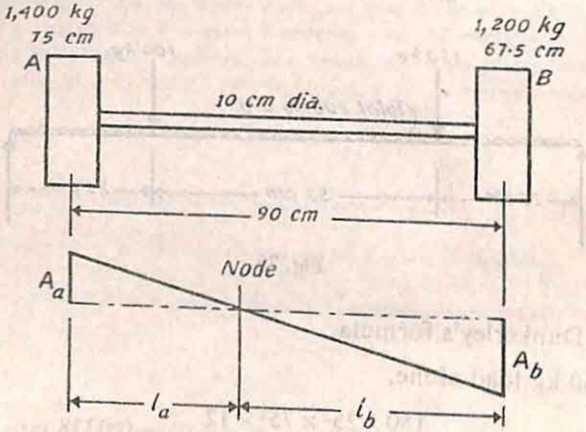

Fig. 76

Let the node divide the shaft into two parts l_a and l_b. Then the frequency of vibration of the system to the left of the node is given by

$$N_a = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_a l_a}}$$

and the frequency of vibration of the system to the right of the node is given by

$$N_b = \frac{1}{2}\sqrt{\frac{CJ}{I_b l_b}}$$

But these two frequencies must be equal.

$$\therefore \quad I_a l_a = I_b l_b$$

or
$$\frac{1{,}400 \times 75^2}{981} \times l_a = \frac{1{,}200 \times 67{\cdot}5^2}{981} \times (90 - l_a)$$

$$\therefore \quad l_a = 36{\cdot}9 \text{ cm}$$

Hence the frequency of vibration

$$= N_a = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_a l_a}} = \frac{1}{2\pi}\sqrt{\frac{0{\cdot}8 \times 10^6 \times \pi \times 10^4 \times 981}{1{,}400 \times 75^2 \times 36{\cdot}9 \times 32}}$$

$$= 8{\cdot}2 \text{ per second}$$

The amplitude ratio

$$\frac{A_a}{A_b} = \frac{l_a}{l_b} = \frac{I_b}{I_a} = 0{\cdot}694.$$

11. The flywheel of an engine driving a dynamo weighs 200 kg and has a radius of gyration of 30 cm. The shaft at the flywheel end has an effective length of 25 cm and is 5 cm diameter. The armature weighs 120 kg and its radius of gyration is 22 cm. The dynamo shaft is 4·5 cm diameter and 20 cm effective length. Calculate the frequency of torsional oscillations and the position of the node. $C = 0{\cdot}8 \times 10^6$ kg/cm². (Lond. Univ.)

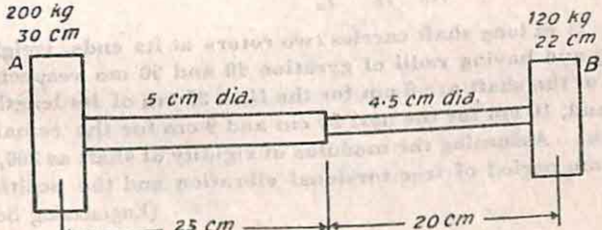

Fig. 77

The equivalent length of shaft 5 cm dia,

$$l = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 = 25 + 20\left(\frac{5}{4{\cdot}5}\right)$$

$$= 55{\cdot}5 \text{ cm}$$

The system Fig. 78 (a) is then torsionally equivalent to the system Fig. 77.

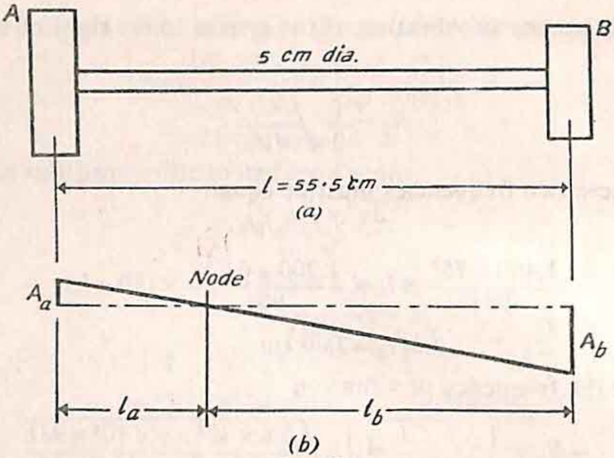

Fig. 78

$$I_a l_a = I_b l_b$$

or
$$\frac{200 \times 30^2}{981} \times l_a = \frac{120 \times 22^2}{981} \times (55{\cdot}5 - l_a)$$

$$\therefore \quad l_a = \mathbf{13{\cdot}54\ cm}$$

Hence frequency of vibration

$$= N_a = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_a l_a}} = \frac{1}{2\pi}\sqrt{\frac{0{\cdot}8 \times 10^6 \times \pi \times 5^4 \times 981}{200 \times 30^2 \times 13{\cdot}54 \times 32}}$$

$$= \mathbf{22{\cdot}4\ per\ second}$$

The node is at a distance of 13·54 cm from the flywheel.

Amplitude ratio $= \frac{A_a}{A_b} = \frac{l_a}{l_b} = \frac{I_b}{I_a} = 0{\cdot}323.$

12. A $1\frac{1}{4}$ m long shaft carries two rotors at its ends, weighing 250 kg and 600 kg and having radii of gyration 40 and 50 cm respectively. The diameters of the shaft are 8 cm for the first 35 cm of its length measured from one end; 10 cm for the next 50 cm and 9 cm for the remaining 40 cm of its length. Assuming the modulus of rigidity of shaft as 800,000 kg/cm², find the time period of free torsional vibration and the position of node point.

(Engineering Services, 1966)

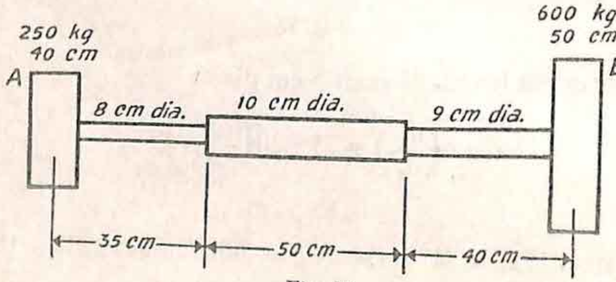

Fig. 79

The equivalent length of shaft of diameter 8 cm,

$$l = 35 + 50(\tfrac{8}{10})^4 + 40(\tfrac{8}{9})^4$$
$$= 35 + 20{\cdot}5 + 25 = 80{\cdot}5 \text{ cm}$$

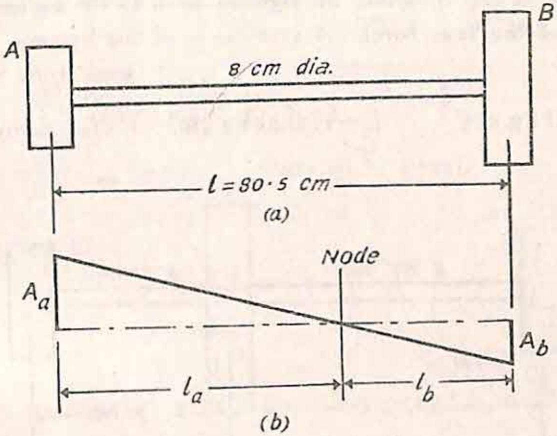

Fig. 80

The system Fig. 80 (a) is then torsionally equivalent to the system Fig. 79. Then the time period of vibration of the system to the left of the node is given by

$$T_a = 2\pi\sqrt{\frac{I_a l_a}{CJ}}$$

and the time period of vibration of the system to the right of the node is given by

$$T_b = 2\pi\sqrt{\frac{I_b l_b}{CJ}}$$

But these two time periods must be equal.

Then $I_a \,.\, l_a = I_b \,.\, l_b$

or
$$\frac{250 \times 40^2}{981} \times l_a = \frac{600 \times 50^2}{981} \times (80{\cdot}5 - l_a)$$

$$\therefore \quad l_a = 63{\cdot}6 \text{ cm and } l_b = 16{\cdot}9 \text{ cm}$$

The node point lies in the part of the shaft with 9 cm diameter.

Distance of node from $B = 16{\cdot}9(\frac{9}{8})^4 = \mathbf{27{\cdot}1 \text{ cm}}$

Time period of vibration

$$= T_a = 2\pi\sqrt{\frac{250 \times 40^2 \times 63{\cdot}6 \times 32}{981 \times 800{,}000 \times \pi \times 8^4}}$$

$$= \mathbf{0{\cdot}0564 \text{ second}}$$

Ratio of amplitude $= \dfrac{A_a}{A_b} = \dfrac{l_a}{l_b} = 3{\cdot}76.$

13. **The moments of inertia of three rotors** A, B **and** C **are respectively 9,000 kg cm², 18,000 kg cm² and 6,000 kg cm². The distance between** A **and** B **is 150 cm and between** B **and** C **is 90 cm and the shaft is 5 cm in diameter. If the modulus of rigidity is** $0{\cdot}8 \times 10^6$ **kg/cm², find the frequencies of the free torsional vibrations of the system.**

(Engineering Services, 1964)

$I_a = 9{,}000$ kg cm² $\qquad I_b = 18{,}000$ kg cm² $\qquad I_c = 6{,}000$ kg cm²

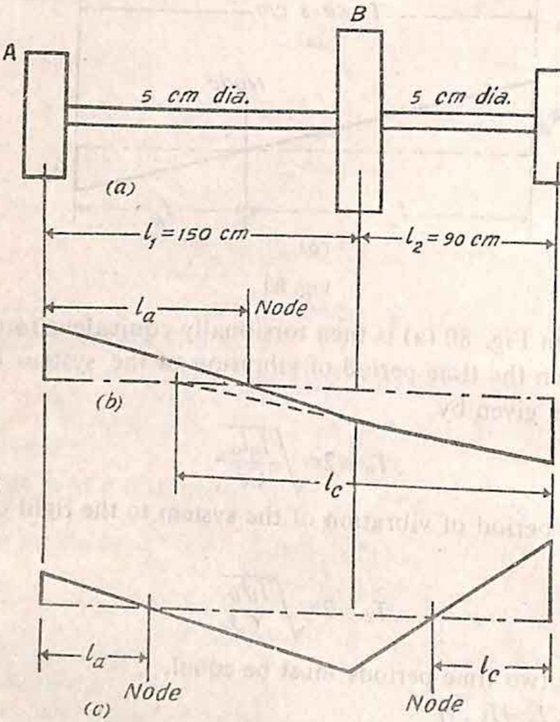

Fig. 81

For the left hand rotor, $N_a = \dfrac{1}{2\pi}\sqrt{\dfrac{K_a}{I_a}} = \dfrac{1}{2\pi}\sqrt{\dfrac{CJ}{I_a l_a}}$

For the right hand rotor, $N_c = \dfrac{1}{2\pi}\sqrt{\dfrac{CJ}{I_c l_c}}$

For the middle rotor, $N_b = \dfrac{1}{2\pi}\sqrt{\dfrac{K_b}{I_b}}$

where K_b, the torque required to twist B through 1 radian when the

shaft is fixed at the nodes, is the sum of the torques required to produce a twist of 1 radian in each of the lengths l_1-l_a and l_2-l_c.

$$\therefore \quad K_b=\frac{CJ}{l_1-l_a}+\frac{CJ}{l_2-l_c}=CJ\left(\frac{1}{l_1-l_a}+\frac{1}{l_2-l_c}\right)$$

Hence $$N_b=\frac{1}{2\pi}\sqrt{\frac{CJ}{I_b}\left(\frac{1}{l_1-l_a}+\frac{1}{l_2-l_c}\right)}$$

The frequencies N_a, N_b and N_c must all be equal.

If we equate the frequency of vibration of I_a to the frequency of vibration of I_c, we have

$$I_a l_a = I_c l_c \qquad \ldots \quad (1)$$

or $$l_c=\frac{9{,}000}{6{,}000}\,l_a=1{\cdot}5 l_a$$

Similarly if we equate the frequencies for I_a and I_b, we have

$$\frac{1}{I_a l_a}=\frac{1}{I_b}\left(\frac{1}{l_1-l_a}+\frac{1}{l_2-l_c}\right)$$

or $$(l_1-l_a)(l_2-l_c)=\frac{I_a}{I_b}\cdot l_a\{l_1+l_2-(l_a+l_c)\} \qquad \ldots \quad (2)$$

Substituting for l_1, l_2, l_c, I_a, and I_b in Eq. (2)

$$(150-l_a)(90-1{\cdot}5l_a)=\frac{9{,}000}{18{,}000}\cdot l_a(240-2{\cdot}5l_a)$$

or $$5{\cdot}5l_a^2-870l_a+27{,}000=0$$

Solving $$l_a=115{\cdot}8 \text{ cm or } 42{\cdot}4 \text{ cm}$$

and $$l_c=1{\cdot}5\, l_a=173{\cdot}7 \text{ cm or } 63{\cdot}6 \text{ cm}$$

The fundamental frequency will be that which corresponds to the larger of these two values of l_a or l_c.

$\therefore$ Fundamental frequency of vibration

$$=N_a=\frac{1}{2\pi}\sqrt{\frac{CJ}{I_a l_a}}=\frac{1}{2\pi}\sqrt{\frac{0{\cdot}8\times10^6\times\pi\times5^4\times981}{9{,}000\times115{\cdot}8\times32}}$$

$$=\mathbf{34{\cdot}2 \text{ per sec.}}$$

This is shown in Fig. b.

The two-node frequency $=34{\cdot}2\sqrt{\dfrac{115{\cdot}8}{42{\cdot}4}}$

$$=\mathbf{56{\cdot}5 \text{ per sec.}}$$

This is shown in Fig. c.

14. A marine engine shaft and propeller are approximately equivalent to the following three-rotor system. The combined moment of inertia of the engine masses is 3,000 kg m^2, that of the flywheel is 9,000 kg m^2, and that of the propeller is 5,400 kg m^2. The equivalent shaft between the engine masses and the flywheel is 38 cm dia. and 5·5 m long and that between the flywheel and the propeller is 35 cm dia. and 11·5 m long. Find, from first principles, the frequencies of the torsional vibrations of the system and the position of the nodes. $C=0{\cdot}8\times10^6$ kg/cm^2.

(Engineering Services, 1959)

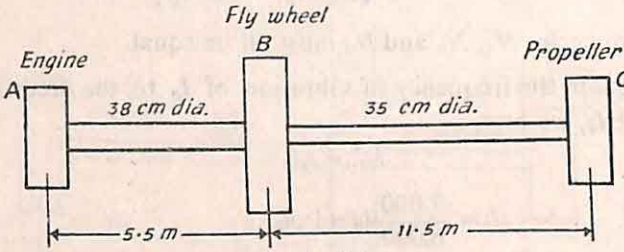

Fig. 82

Reduce the system to an equivalent shaft of 35 cm diameter. The length l_1 of the torsionally equivalent shaft of 35 cm dia. between the engine and the flywheel

$$=5{\cdot}5\left(\frac{35}{38}\right)^4=3{\cdot}96 \text{ m}$$

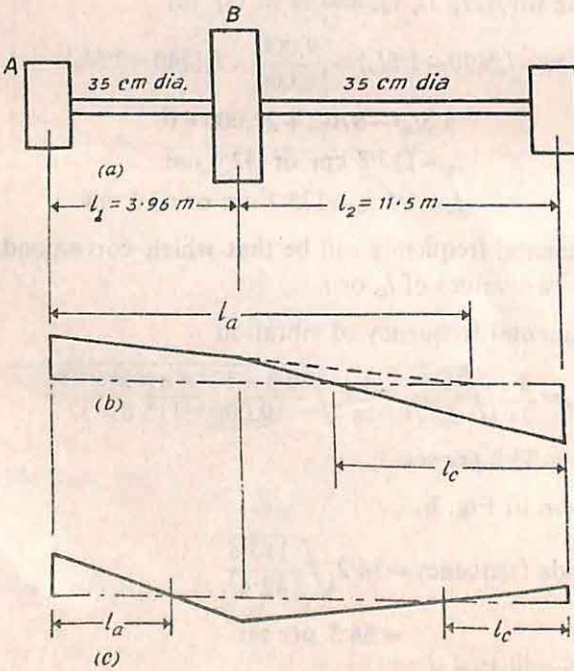

Fig. 83

Fig. 83 (a) shows a three-rotor system on 35 cm dia. shaft throughout.

As in the previous example

$$I_a l_a = I_c l_c \qquad .. \qquad (1)$$

or

$$l_a = \frac{5,400}{3,000} l_c = 1{\cdot}8\ l_c$$

and

$$\frac{1}{I_a l_a} = \frac{1}{I_b}\left(\frac{1}{l_1 - l_a} + \frac{1}{l_2 - l_c}\right)$$

or

$$(l_1 - l_a)(l_2 - l_c) = \frac{I_a}{I_b} . l_a\{l_1 + l_2 - (l_a + l_c)\} \qquad .. \qquad (2)$$

or

$$(3{\cdot}96 - 1{\cdot}8l_c)(11{\cdot}5 - l_c) = \frac{3,000}{9,000} \times 1{\cdot}8l_c(15{\cdot}46 - 2{\cdot}8l_c)$$

or

$$3{\cdot}48l_c^2 - 33{\cdot}98l_c + 45{\cdot}5 = 0$$

or

$$l_c^2 - 9{\cdot}76l_c + 13{\cdot}07 = 0$$

Solving $\quad l_c = 8{\cdot}16\ m$ or $1{\cdot}60\ m$

and $\quad l_a = 1{\cdot}8l_c = 14{\cdot}69\ m$ or $2{\cdot}88\ m$

Fundamental frequency of vibration

$$= N_c = \frac{1}{2\pi}\sqrt{\frac{CJ}{I_c l_c}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{0{\cdot}8 \times 10^6 \times \pi \times 35^4 \times 981}{5,400 \times (100)^2 \times 8{\cdot}16 \times 100 \times 32}}$$

$$= \mathbf{8{\cdot}15 \text{ per sec.}}$$

The one node occurs at **8·16 m** from the propeller. This is shown in Fig. 83 (b).

The two-node frequency $= 8{\cdot}15\sqrt{\dfrac{8{\cdot}16}{1{\cdot}60}} = \mathbf{18{\cdot}4 \text{ per sec.}}$

This is shown in Fig. 83 (c).

$$l_a = 2{\cdot}88\ m,\ l_c = \mathbf{1{\cdot}60\ m}$$

Now l_a on the 38 cm dia. will be

$$= 2{\cdot}88\left(\frac{38}{35}\right)^4 = \mathbf{4\ m}$$

Hence one node occurs at 4 m from the engine and the other at 1·60 m from the propeller.

15. A light elastic shaft AB of uniform diameter, supported freely in bearings, carries a wheel at each end, and it is found that the natural frequency of torsional vibration is 40 per second. A third wheel is mounted on the shaft at a point *C* such that $AC=\frac{3}{4}AB$. If all the wheels have the same moment of inertia, determine the natural frequencies of torsional vibration. (Lond. Univ.)

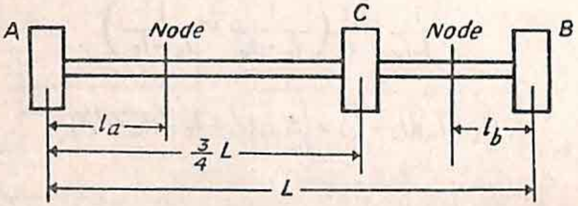

Fig. 84

Let I be the moment of inertia of each wheel.

With two wheels at the two ends the node is at the centre of the span L.

Hence
$$N=\frac{1}{2\pi}\sqrt{\frac{2CJ}{IL}}$$

or
$$40=\frac{1}{2\pi}\sqrt{\frac{2CJ}{IL}}$$

$$\therefore \quad \frac{CJ}{IL}=3{,}200\pi^2$$

With three equal wheels let the nodes be at distances l_a and l_b from the ends A and B respectively.

For the left hand rotor :

$$N_a=\frac{1}{2\pi}\sqrt{\frac{CJ}{Il_a}}$$

For the right hand rotor :

$$N_b=\frac{1}{2\pi}\sqrt{\frac{CJ}{Il_b}}$$

For the middle rotor :

$$N_c=\frac{1}{2\pi}\sqrt{\frac{CJ}{I}\left\{\frac{1}{\frac{3}{4}L-l_a}+\frac{1}{\frac{L}{4}-l_b}\right\}}$$

Equating N_a to N_b, we have $l_a=l_b$ .. (1)

Equating N_a to N_c, we get

$$\frac{1}{l_a}=\frac{1}{\frac{3}{4}L-l_a}+\frac{1}{\frac{L}{4}-l_b}$$

or $$\left(\frac{3}{4}L-l_a\right)\left(\frac{L}{4}-l_b\right)=l_a(L-l_a-l_b) \quad .. \quad (2)$$

Substituting l_b from eq. 1

$$\left(\frac{3}{4}L-l_a\right)\left(\frac{L}{4}-l_a\right)=l_a(L-2l_a)$$

or $$3l_a^2-2Ll_a+\frac{3L^2}{16}=0$$

Solving $\qquad l_a=0{\cdot}554L \quad \text{or} \quad 0{\cdot}113L$

Hence frequency of one-node vibration

$$=N_a=\frac{1}{2\pi}\sqrt{\frac{CJ}{I\times 0{\cdot}554L}}=\frac{1}{2\pi}\sqrt{\frac{3200\pi^2}{0{\cdot}554}}$$

$$=\mathbf{38\ per\ second}$$

and the frequency of two-node vibration

$$=N_a=\frac{1}{2\pi}\sqrt{\frac{CJ}{I\times 0{\cdot}113\,L}}=\frac{1}{2\pi}\sqrt{\frac{3200\pi^2}{0{\cdot}113}}$$

$$=\mathbf{84{\cdot}14\ per\ second.}$$

16. A shaft 10 cm diameter is supported in bearings 2·5 m apart. It carries two pulleys which weigh 200 kg and 150 kg at distances of 1 m and 2 m respectively from one bearing. Calculate the whirling speed by (a) Dunkerley's method, (b) the energy method. $E=2\times10^6$ kg/cm².

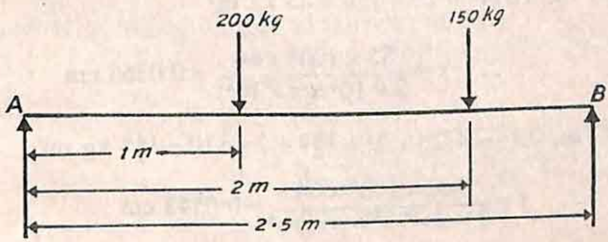

Fig. 85

(a) Dunkerley's method :

For 200 kg load alone

$$\delta_1=\frac{200\times100^2\times150^2\times64}{3\times2\times10^6\times\pi\times10^4\times250}=0{\cdot}0611\ \text{cm}$$

For 150 kg. load alone

$$\delta_2 = \frac{150 \times 200^2 \times 50^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 10^4 \times 250} = 0{\cdot}0204 \text{ cm}$$

$$N = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}0815}}$$

$$= \mathbf{17{\cdot}46 \text{ per sec.}}$$

(b) Energy method :

$$R_B \times 2{\cdot}5 = 200 \times 1 + 150 \times 2$$

$$\therefore \quad R_B = 200 \text{ kg and } R_A = 150 \text{ kg}$$

$$M = 150x - 200[x-1] - 150[x-2]$$

$$EI\,\frac{d^2y}{dx^2} = -M$$

$$= -150x + 200[x-1] + 150[x-2]$$

Integrating

$$EI\,\frac{dy}{dx} = -75x^2 + 100[x-1]^2 + 75[x-2]^2 + A$$

Integrating again

$$EIy = -25x^3 + \tfrac{100}{3}[x-1]^3 + 25[x-2]^3 + Ax + B$$

At $x=0$, $y=0$ $\quad\therefore\quad B=0$

At $x=2{\cdot}5$ m, $y=0$

$$\therefore \quad 0 = -25 \times 2{\cdot}5^3 + \tfrac{100}{3} \times 1{\cdot}5^3 + 25 \times 0{\cdot}5^3 + 2{\cdot}5A$$

$$\therefore \quad A = 110 \text{ kg m}^2$$

At $x = 1\ m$, $EIy = -25 + 110 = 85$ kg m^3

$$\therefore \quad y = \frac{85 \times 100^3 \times 64}{2 \times 10^6 \times \pi \times 10^4} = 0{\cdot}0866 \text{ cm}$$

At $x=2\ m$, $EIy = -25 \times 2^3 + \tfrac{100}{3} + 2 \times 110 = \tfrac{160}{3}$ kg m^3

$$\therefore \quad y = \frac{160 \times 100^3 \times 64}{3 \times 2 \times 10^6 \times \pi \times 10^4} = 0{\cdot}0543 \text{ cm}$$

$$N = \frac{1}{2\pi}\sqrt{g.\frac{\Sigma Wy}{\Sigma Wy^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{981 \times (200 \times 0{\cdot}0866 + 150 \times 0{\cdot}0543)}{(200 \times 0{\cdot}0866^2 + 150 \times 0{\cdot}0543^2)}}$$

$$= \mathbf{18{\cdot}05 \text{ per sec.}}$$

17. Calculate the whirling speed of a steel shaft 5 cm diameter and 3 m long, carrying a wheel weighing 30 kg at 60 cm from one end and one weighing 20 kg at 90 cm from the other. The shaft may be considered as simply supported on bearings at the ends. Density of steel 7·8 g/cm³. $E=2\times10^6$ kg/cm². (Lond. Univ.)

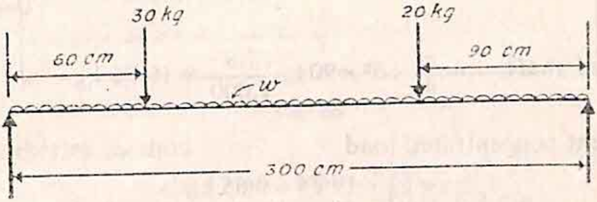

Fig. 86

Apply Dunkerley's method.

For 30 kg load alone

$$\delta_1=\frac{30\times60^2\times240^2\times64}{3\times2\times10^6\times\pi\times5^4\times300}=0{\cdot}1127\text{ cm}$$

For 20 kg load alone

$$\delta_2=\frac{20\times210^2\times90^2\times64}{3\times2\times10^6\times\pi\times5^4\times300}=0{\cdot}1293\text{ cm}$$

Considering concentrated loads only

$$N_o=\frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}}=\frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}242}}$$

$$=\mathbf{10{\cdot}14\ per\ sec.}$$

Weight of shaft per cm length

$$w=\frac{\pi}{4}\times5^2\times\frac{7{\cdot}8}{1{,}000}=0{\cdot}153\text{ kg}$$

Frequency of vibration due to self weight only

$$N_s=\frac{\pi}{2l^2}\sqrt{\frac{gEI}{w}}$$

$$=\frac{\pi}{2\times300^2}\sqrt{\frac{981\times2\times10^6\times\pi\times5^4}{0{\cdot}153\times64}}$$

$$=\mathbf{10{\cdot}95\ per\ sec.}$$

$$\frac{1}{N^2}=\frac{1}{{N_o}^2}+\frac{1}{{N_s}^2}$$

$$=\frac{1}{10{\cdot}14^2}+\frac{1}{10{\cdot}95^2}$$

$$\therefore\quad N=\mathbf{7{\cdot}44\ per\ sec.}$$

18. A steel shaft, 6 cm diameter, with its ends freely supported in bearings 90 cm apart, carries two wheels, each weighing 50 kg and each placed 30 cm from one of the bearings. Calculate the lowest whirling speed of the shaft, assuming the shaft effect to be equivalent to $\frac{17}{35}$ of its mass concentrated at mid-span. Take $\rho = 7{\cdot}8$ g/cm³ and $E = 2 \times 10^6$ kg/cm² for steel.

(Lond. Univ.)

$$\text{Weight of shaft} \quad = \frac{\pi}{4} \times 6^2 \times 90 \times \frac{7{\cdot}8}{1{,}000} = 19{\cdot}86 \text{ kg}$$

Equivalent concentrated load

$$= \tfrac{17}{35} \times 19{\cdot}86 = 9{\cdot}65 \text{ kg}$$

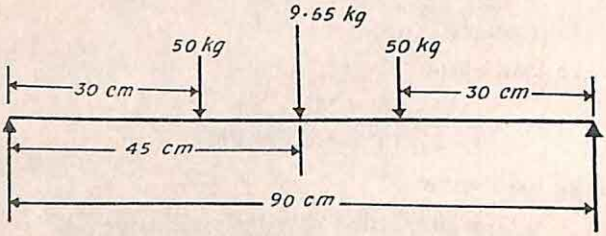

Fig. 87

For 50 kg load alone,

$$\delta_1 = \frac{50 \times 30^2 \times 60^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 6^4 \times 90} = 0{\cdot}00472 \text{ cm}$$

For 9·65 kg load alone

$$\delta_2 = \frac{9{\cdot}65 \times 90^3 \times 64}{48 \times 2 \times 10^6 \times \pi \times 6^4} = 0{\cdot}00115 \text{ cm}$$

Apply Dunkerley's formula

$$N = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}01059}}$$

$$= 48{\cdot}4 \text{ per second.}$$

19. A small turbine rotor has a shaft of uniform section, $EI = 1{,}200 \times 10^6$ kg cm², and is freely supported in two bearings at 1 metre centres. It carries three equal wheels, 400 kg each, at positions 25 cm, 40 cm, and 55 cm from one bearing. The static deflections at the wheels, neglecting shaft weight, are estimated respectively as 0·0133 cm, 0·0176 cm, and 0·0178 cm. Compare the critical speeds as calculated (a) by Dunkerley method, (b) by the energy formula.

(Lond. Univ.)

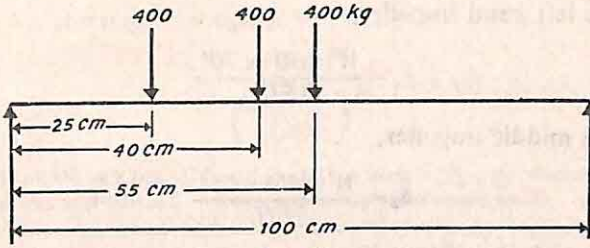

Fig. 88

(a) Dunkerley method :

$$\delta_1 = \frac{400 \times 25^2 \times 75^2}{3 \times 1{,}200 \times 10^6 \times 100} = 0{\cdot}00391 \text{ cm}$$

$$\delta_2 = \frac{400 \times 40^2 \times 60^2}{3 \times 1{,}200 \times 10^6 \times 100} = 0{\cdot}00640 \text{ cm}$$

$$\delta_3 = \frac{400 \times 55^2 \times 45^2}{3 \times 1{,}200 \times 10^6 \times 100} = 0{\cdot}00681 \text{ cm}$$

$$N = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{0{\cdot}01712}}$$

$$= \mathbf{38{\cdot}1 \text{ per second.}}$$

(b) Energy method :

$$N = \frac{1}{2\pi}\sqrt{g.\frac{\Sigma WY}{\Sigma WY^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{981 \times 400(0{\cdot}0133 + 0{\cdot}0176 + 0{\cdot}0178)}{400(0{\cdot}0133^2 + 0{\cdot}0176^2 + 0{\cdot}0178^2)}}$$

$$= \mathbf{38{\cdot}8 \text{ per second.}}$$

20. The rotor of a three-stage pump is carried in spherically seated bearings at 100 cm centres. The three impellers weigh 350 kg each and are at distances of 30 cm, 45 cm and 60 cm from one bearing centre line. The critical speed must be kept above 2,500 r.p.m. Estimate the minimum permissible shaft diameter, neglecting the mass of the shaft itself. Take E for the material of the shaft as 2×10^6 kg/cm². (Engineering Services, 1962)

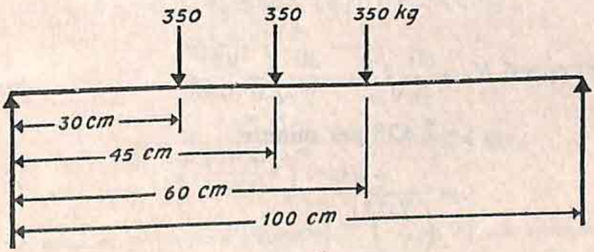

Fig. 89

For the left hand impeller,

$$\delta_1 = \frac{W \times 30^2 \times 70^2}{3EIL}$$

For the middle impeller,

$$\delta_2 = \frac{W \times 45^2 \times 55^2}{3EIL}$$

For the right hand impeller,

$$\delta_3 = \frac{W \times 60^2 \times 40^2}{3EIL}$$

$$\Sigma\delta = \frac{W}{3EIL}(30^2 \times 70^2 + 45^2 \times 55^2 + 60^2 \times 40^2)$$

$$= \frac{350 \times 64 \times 1{,}630 \times 10^4}{3 \times 2 \times 10^6 \times \pi d^4 \times 100}$$

$$= \frac{56 \times 163}{15\pi d^4}$$

$$N = \frac{1}{2\pi}\sqrt{\frac{g}{\Sigma\delta}} = \frac{1}{2\pi}\sqrt{\frac{981 \times 15\pi d^4}{56 \times 163}}$$

$$= 0{\cdot}358 d^2 \text{ per sec.}$$

or
$$\frac{2{,}500}{60} = 0{\cdot}358 d^2$$

$$\therefore \quad d = \mathbf{10{\cdot}78 \text{ cm.}}$$

21. A shaft 2 cm diameter is freely supported in bearings 30 cm apart and carries a wheel which weighs 60 kg at a point 12 cm from one bearing. E for shaft $= 2 \times 10^6$ kg/cm². Find the whirling speed.

If the wheel is mounted on the shaft with its centre of mass initially 0·12 cm out of alignment with the shaft axis, find the deflection of the shaft when its speed is 2,500 r.p.m.

$$\delta = \frac{60 \times 12^2 \times 18^2 \times 64}{3 \times 2 \times 10^6 \times \pi \times 2^4 \times 30} = 0{\cdot}0198 \text{ cm}$$

$$\text{Critical speed } N_c = \frac{60}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{30}{\pi}\sqrt{\frac{981}{0{\cdot}0198}}$$

$$= \mathbf{2{,}125 \text{ per minute}}$$

$$y = \frac{\pm e}{\left(\frac{N_c}{N}\right)^2 - 1}$$

When $N > N_c$, the negative sign is to be taken.

$$y = \frac{-0{\cdot}12}{\left(\frac{2{,}125}{2{,}500}\right)^2 - 1} = \mathbf{0{\cdot}432\ cm.}$$

22. The shaft of a small turbine with a single disc is observed to have a static deflection of 0·038 cm. Determine the whirling speed. Also calculate the required percentage change in diameter of the shaft to raise the whirling speed to 2,100 r.p.m.

For the above two cases calculate the deflections of the shaft at a speed of 1,400 r.p.m., if the initial displacement of the centre of mass of the disc from the axis of the shaft is 0·053 cm. (Engineering Services, 1961)

Whirling speed, $N_1 = \frac{60}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{30}{\pi}\sqrt{\frac{981}{0{\cdot}038}}$

$$= \mathbf{1{,}534\ r.p.m.}$$

$$\frac{N_1}{N_2} = \sqrt{\frac{\delta_2}{\delta_1}} = \sqrt{\frac{I_1}{I_2}} = \frac{d_1^2}{d_2^2}$$

$$\therefore \quad \frac{d_2}{d_1} = \sqrt{\frac{N_2}{N_1}} = \sqrt{\frac{2{,}100}{1{,}534}} = 1{\cdot}17$$

Percentage change in dia.

$$= \frac{d_2 - d_1}{d_1} \times 100 = (1{\cdot}17 - 1) \times 100$$

$$= \mathbf{17\ increase}$$

For an eccentric disc

$$y = \frac{\pm e}{\left(\frac{N_c}{N}\right)^2 - 1}$$

When $N < N_c$ the positive sign is to be taken.

1st case :

$$y = \frac{0{\cdot}053}{\left(\frac{1{,}534}{1{,}400}\right)^2 - 1} = \mathbf{0{\cdot}264\ cm}$$

2nd case :

$$y = \frac{0{\cdot}053}{\left(\frac{2{,}100}{1{,}400}\right)^2 - 1} = \mathbf{0{\cdot}0424\ cm.}$$

23. A vertical steel shaft 6 mm diameter and 20 cm long is supported in long bearings at its ends. It carries a wheel weighing 2 kg at the centre

of the shaft. Find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 80% of the critical speed. The centre of gravity of the wheel is 0·02 cm from the centre of the shaft. $E=2\times10^6$ kg/cm².

$$=\frac{WL^3}{192EI}=\frac{2\times20^3\times64}{192\times2\times10^6\times\pi\times(0{\cdot}6)^4}$$

$$=0{\cdot}00655 \text{ cm}$$

$$\text{Critical speed } N_c=\frac{60}{2\pi}\sqrt{\frac{981}{0{\cdot}00655}}$$

$$=\mathbf{3{,}700 \text{ r.p.m.}}$$

$$\frac{y}{e}=\frac{1}{\left(\frac{N_c}{N}\right)^2-1}$$

$$\text{At } N=0{\cdot}8N_c,\quad y=\frac{0{\cdot}02}{\left(\frac{1}{0{\cdot}8}\right)^2-1}$$

$$=0{\cdot}0356 \text{ cm}$$

∴ Central (centrifugal) bending force

$$=\frac{192EI}{L^3}\cdot y=\frac{192\times2\times10^6\times\pi(0{\cdot}6)^4\times0{\cdot}0356}{20^3\times64}$$

$$=10{\cdot}87 \text{ kg}$$

Bending moment,

$$M=\frac{WL}{8}=\frac{10{\cdot}87\times20}{8}=\mathbf{27{\cdot}2 \text{ kg cm}}$$

Maximum bending stress,

$$f=\frac{M}{Z}=\frac{27{\cdot}2\times32}{\pi\times(0{\cdot}6)^3}=\mathbf{1{,}282 \text{ kg/cm}^2}.$$

24. Solve Example 23 if the shaft bearings do not fix its direction at the ends.

$$\delta=\frac{WL^3}{48EI}=\frac{2\times20^3\times64}{48\times2\times10^6\times\pi\times(0{\cdot}6)^4}$$

$$=0{\cdot}0262 \text{ cm}$$

$$N_c=\frac{60}{2\pi}\sqrt{\frac{981}{0{\cdot}0262}}=\mathbf{1{,}848 \text{ r.p.m.}}$$

At $N=0{\cdot}8\ N_c$ $y=\dfrac{0{\cdot}02}{\left(\dfrac{1}{0{\cdot}8}\right)^2-1}$

$=0{\cdot}0356$ cm

Equivalent central load

$$=\frac{48EI}{L^3}.\,y=\frac{48\times 2\times 10^6\times\pi\times(0{\cdot}6)^4\times 0{\cdot}0356}{20^3\times 64}$$

$=2{\cdot}72$ kg

Maximum bending moment,

$$M=\frac{WL}{4}=\frac{2{\cdot}72\times 20}{4}=13{\cdot}6 \text{ kg cm}$$

Maximum stress,

$$f=\frac{M}{Z}=\frac{13{\cdot}6\times 32}{\pi\times(0{\cdot}6)^3}=\mathbf{641\ kg/cm^2}.$$

25. A shaft 1 cm diameter rotates in spherical bearings with a span of 80 cm and carries a disc of mass 5 kg midway between the two bearings. The mass centre of the disc is 0·02 cm out of centre. Neglecting the mass of the shaft determine the critical speed.

If the stress in the shaft is not to exceed 800 kg/cm², determine the range of speed within which it is unsafe to run the shaft. $E=2\times 10^6$ kg/cm². (Lond. Univ.)

Static deflection, $\delta=\dfrac{WL^3}{48EI}$

$$=\frac{5\times 80^3\times 64}{48\times 2\times 10^6\times\pi}=0{\cdot}543 \text{ cm}$$

Critical speed $N_c=\dfrac{60}{2\pi}\sqrt{\dfrac{981}{0{\cdot}543}}=\mathbf{406\ r.p.m.}$

Let P be the dynamic load to cause 800 kg/cm² stress.

Then bending moment $M=\dfrac{P\times 80}{4}=20P$

Stress $f=\dfrac{M}{Z}$

or $800=\dfrac{20P\times 32}{\pi}$

$\therefore\quad P=\dfrac{5\pi}{4}$ kg

Deflection y due to $P=\frac{5\pi}{4\times 5}\times 0{\cdot}543=0{\cdot}426$ cm

$$\frac{y}{e}=\frac{\pm 1}{\left(\frac{N_c}{N}\right)^2-1}$$

or $$\frac{0{\cdot}426}{0{\cdot}02}=\frac{\pm 1}{\left(\frac{N_c}{N}\right)^2-1}$$

or $$\left(\frac{N_c}{N}\right)^2=1\pm\frac{0{\cdot}02}{0{\cdot}426}$$

$$=1{\cdot}0469 \quad \text{or} \quad 0{\cdot}9531$$

$\therefore$ $$\frac{N_c}{N}=1{\cdot}023 \quad \text{or} \quad 0{\cdot}976$$

$\therefore$ $$N=\frac{406}{1{\cdot}023} \quad \text{or} \quad \frac{406}{0{\cdot}976}$$

$$=397 \quad \text{or} \quad 416 \text{ per minute}$$

The range of speed is between **397 r.p.m.** and **416 r.p.m.**

26. A shaft 12 mm diameter rotates in long fixed bearings and a disc of mass 20 kg is secured to the shaft at the middle of its length. The span of the shaft between the bearings is 60 cm. The mass centre of the disc is 0·05 cm from the axis of the shaft. Neglecting the mass of the shaft, determine its critical speed. Young's modulus $E=2\times 10^6$ kg/cm².

If the stress in the shaft due to bending is not to be greater than 1,200 kg/cm², find the range of speed over which this stress would be exceeded.

(Lond. Univ.)

$$\delta=\frac{WL^3}{192EI}=\frac{20\times 60^3\times 64}{192\times 2\times 10^6\times\pi\times(1{\cdot}2)^4}$$

$$=0{\cdot}1105 \text{ cm}$$

$$N_c=\frac{60}{2\pi}\sqrt{\frac{981}{0{\cdot}1105}}=\mathbf{900\ r.p.m.}$$

$$M=\frac{PL}{8}=\frac{P\times 60}{8}=7{\cdot}5\,P$$

$$f=\frac{M}{Z}$$

or $$1{,}200=\frac{7{\cdot}5\,P\times 32}{\pi\times(1{\cdot}2)^3}$$

$\therefore$ $$P=27{\cdot}1 \text{ kg}$$

Deflection y due to $P = \dfrac{27{\cdot}1}{20} \times 0{\cdot}1105$

$$= 0{\cdot}15 \text{ cm}$$

$$\frac{y}{e} = \frac{\pm 1}{\left(\dfrac{N_c}{N}\right)^2 - 1}$$

or
$$\frac{0{\cdot}15}{0{\cdot}05} = \frac{\pm 1}{\left(\dfrac{N_c}{N}\right)^2 - 1}$$

or
$$\left(\frac{N_c}{N}\right)^2 = 1 \pm \frac{1}{3}$$

or
$$\frac{N_c}{N} = \sqrt{\frac{4}{3}} \quad \text{or} \quad \sqrt{\frac{2}{3}}$$

$\therefore$
$$N = 900 \times \sqrt{\frac{3}{4}} \quad \text{or} \quad 900\sqrt{\frac{2}{3}}$$

$$= 779 \quad \text{or} \quad 1{,}102 \text{ r.p.m.}$$

Hence the range of speed is between **779 r.p.m.** and **1,102 r.p.m.**

CHAPTER IX

THEORIES OF FAILURE

1. A piece of material is subjected to three mutually perpendicular tensile stresses of 500, 650 and 800 kg/cm². Calculate the strain energy per unit volume. Calculate also the maximum shear strain energy per unit volume. Poisson's ratio=0·3; $E=2\times10^6$ kg/cm². (Lond. Univ.)

$f_1=800$ kg/cm², $f_2=650$ kg/cm², $f_3=500$ kg/cm²

Total strain energy per unit volume

$$U=\frac{1}{2E}\left[f_1^2+f_2^2+f_3^2-\frac{2}{m}\left(f_1f_2+f_2f_3+f_3f_1\right)\right]$$

$$=\frac{1}{4\times10^6}\left[800^2+650^2+500^2-2\times0\cdot3(800\times650+650\times500+500\times800)\right]$$

$$=\mathbf{0\cdot1414\ kg\ cm/cm^3}$$

Maximum shear strain energy

$$U_s=\frac{1}{12C}\left[(f_1-f_2)^2+(f_2-f_3)^2+(f_3-f_1)^2\right]$$

But $$E=2C\left(1+\frac{1}{m}\right)$$

or $$2\times10^6=2C(1+0\cdot3)$$

$\therefore$ $$C=\frac{10^6}{1\cdot3}$$

$$U_s=\frac{1\cdot3}{12\times10^6}\left[150^2+150^2+300^2\right]$$

$$=\mathbf{0\cdot0146\ kg\ cm/cm^3}.$$

2. At a point in a steel member the major principal stress is 2,000 kg/cm² and the minor principal stress is compressive. If the tensile yield point of the steel is 2,500 kg/cm², find the value of the minor principal stress at which yielding will commence, according to each of the following criteria of failure: (a) maximum shearing stress; (b) maximum total strain energy; (c) maximum shear strain energy. Poisson's ratio=0·28.

(Lond. Univ.)

(a) Maximum shearing stress theory :

$$\frac{1}{2}\left(f_1-f_2\right)=\frac{f}{2}$$

or $$f_1-f_2=f$$
or $$2{,}000-f_2=2{,}500$$
$\therefore$ $$f_2=-500 \text{ kg/cm}^2$$

(b) Maximum total strain energy theory :

$$\frac{1}{2E}\left[f_1^2+f_2^2+f_3^2-\frac{2}{m}\left(f_1f_2+f_2f_3+f_3f_1\right)\right]=\frac{f^2}{2E}$$

Putting $f_3=0$, we get

$$f_1^2+f_2^2-\frac{2}{m}f_1f_2=f^2$$

or $$2{,}000^2+f_2^2-2\times0{\cdot}28\times2{,}000\,f_2=2{,}500^2$$
or $$f_2^2-1{,}120\,f_2-2{,}250{,}000=0$$
Solving $$f_2=-1{,}041 \text{ kg/cm}^2.$$

(c) Maximum shear strain energy theory :

$$\frac{1}{12C}[(f_1-f_2)^2+(f_2-f_3)^2+(f_3-f_1)^2]=\frac{f^2}{6C}$$

Putting $f_3=0$, we get

$$f_1^2+f_2^2-f_1f_2=f^2$$
or $$2{,}000^2+f_2^2-2{,}000f_2=2{,}500^2$$
or $$f_2^2-2{.}000f_2-2{,}250{,}000=0$$
Solving $$f_2=-803 \text{ kg/cm}^2.$$

3. At a point in a stressed material the direct stresses on two perpendicular planes are 1,400 kg/cm² tension and 900 kg/cm² compression respectively, and the shearing stress on these planes is q. The yield stress for the material is 2,500 kg/cm². Find the value of q at which failure may be expected, according to each of the following theories of failure: (a) maximum principal stress theory; (b) maximum shearing stress theory; (c) maximum shear strain energy theory. (Lond. Univ.)

Principal stresses

$$=\tfrac{1}{2}(f_x+f_y)\pm\tfrac{1}{2}\sqrt{(f_x-f_y)^2+4q^2}$$
$$=\tfrac{1}{2}(1{,}400-900)\pm\tfrac{1}{2}\sqrt{(1{,}400+900)^2+4q^2}$$
$$=250\pm\tfrac{1}{2}\sqrt{2{,}300^2+4q^2}$$

i.e., $$f_1=250+\tfrac{1}{2}\sqrt{2{,}300^2+4q^2}$$
and $$f_2=250-\tfrac{1}{2}\sqrt{2{,}300^2+4q^2}$$

(a) Maximum prinoipal stress theory :

$$f_1=f$$
or $$250+\tfrac{1}{2}\sqrt{2{,}300^2+4q^2}=2{,}500$$
or $$2{,}300^2+4q^2=4{,}500^2$$
$\therefore$ $$q=1{,}934 \text{ kg/cm}^2.$$

(b) Maximum shearing stress theory :
f_2 is compressive.

$\therefore \quad \frac{1}{2}(f_1-f_2)=\frac{1}{2}f$

or $\quad f_1-f_2=f$

or $\quad \sqrt{2,300^2+4q^2}=2,500$

$\therefore \quad q=490$ **kg/cm²**

(c) Maximum shear strain energy theory :

$$f_1^2+f_2^2-f_1 f_2=f^2$$

or

$$[250+\tfrac{1}{2}\sqrt{2,300^2+4q^2}]^2+[250-\tfrac{1}{2}\sqrt{2,300^2+4q^2}]^2$$
$$-[250+\tfrac{1}{2}\sqrt{[2,300^2+4q^2]}\times[250-\tfrac{1}{2}\sqrt{2,300^2+4q^2}]=2,500^2$$

or $\quad 250^2+\frac{3}{4}(2,300^2+4q^2)=2,500^2$

or $\quad q^2=740,000$

$\therefore \quad q=860$ **kg/cm².**

4. In a two-dimensional stress system, normal stresses of 200 and 1,200 kg/cm² act on two mutually perpendicular planes in conjunction with a shear stress of 400 kg/cm². The stress intensity, judged by the shear strain energy, is excessive. As it was found impossible to reduce the applied stresses, the severity of the shear strain energy condition was reduced by increasing the normal stress of 200 kg/cm² to some higher tensile value, X. Find the value of X at which the shear strain energy is least.

(Lond. Univ.)

Principal stresses

$$=\tfrac{1}{2}(1,200+X)\pm\tfrac{1}{2}\sqrt{(1,200-X)^2+4\times400^2}$$

i.e., $\quad f_1=\frac{1}{2}(1,200+X)+\frac{1}{2}\sqrt{(1,200-X)^2+4\times400^2}$

and $\quad f_2=\frac{1}{2}(1,200+X)-\frac{1}{2}\sqrt{(1,200-X)^2+4\times400^2}$

Shear strain energy,

$$U_s=\frac{1}{12C}[(f_1-f_2)^2+(f_2-f_3)^2+(f_3-f_1)^2]$$

$$=\frac{1}{6C}(f_1^2+f_2^2-f_1 f_2),\ \text{putting } f_3=0$$

$$=\frac{1}{6C}[\tfrac{1}{4}(1,200+X)^2+\tfrac{3}{4}\{(1,200-X)^2+4\times400^2\}]$$

$$=\frac{1}{6C}[X^2-1,200X+1\,920,000]$$

For minimum shear strain energy

$$\frac{dU_s}{dX}=0$$

or
$$2X-1{,}200=0$$
$$\therefore \quad X=600 \text{ kg/cm}^2.$$

5. A solid circular shaft is required to carry a twisting moment of 60,000 kg cm and a bending moment of 20,000 kg cm. Determine the diameter of the shaft on the assumption that the maximum total strain energy per unit volume is not to exceed that in material under a pure shearing stress of 300 kg/cm². $E=2\times10^6$ kg/cm² and Poisson's ratio$=\frac{1}{3{\cdot}5}$

(Lond. Univ.)

$$\text{Principal stresses}=\frac{32}{\pi d^3}\times\frac{1}{2}\left[M\pm\sqrt{M^2+T^2}\right]$$

$$=\frac{16}{\pi d^3}\left[20{,}000\pm\sqrt{20{,}000^2+60{,}000^2}\right]$$

$$=\frac{16}{\pi d^3}\left[20{,}000\pm63{,}250\right]$$

i.e., $$f_1=\frac{16}{\pi d^3}\times83{,}250=\frac{1{,}332{,}000}{\pi d^3}$$

and $$f_2=-\frac{16}{\pi d^3}\times43{,}250=\frac{-692{,}000}{\pi d^3}$$

Maximum total strain energy

$$=\frac{1}{2E}\left(f_1^2+f_2^2-\frac{2}{m}f_1f_2\right)$$

$$=\frac{1}{2E}\times\frac{\left(1{,}332{,}000^2+692{,}000^2+2\times\frac{1}{3.5}\times1{,}332{,}000\times692{,}000\right)}{\pi^2d^6}$$

$$=\frac{1}{2E}\times\frac{278\times10^{10}}{\pi^2d^6}$$

Shear strain energy due to shearing stress

$$=\frac{q^2}{2C}=\frac{300^2}{2C}$$

Hence $$\frac{1}{2E}\times\frac{278\times10^{10}}{\pi^2d^6}=\frac{300^2}{2C}$$

or $$d^6=\frac{C}{E}\times\frac{278\times10^{10}}{\pi^2\times300^2}$$

$$=\frac{278\times10^{10}}{2\left(1+\frac{1}{3\cdot5}\right)\times\pi^2\times300^2}$$

$$\therefore\quad d=10\cdot33\text{ cm.}$$

6. A cylindrical shaft, 7·5 cm in diameter, is subjected to a maximum bending moment of 250 kg m and a twisting moment of 420 kg m. Find the maximum principal stress developed in the shaft. If the yield stress of the shaft material is 3,800 kg/cm², determine the factor of safety of the shaft according to the maximum shearing stress theory of failure.

(Engineering Services, 1967)

Principal stresses

$$=\frac{32}{\pi d^3}\times\frac{1}{2}\left[M\pm\sqrt{M^2+T^2}\right]$$

$$=\frac{16}{\pi\times7\cdot5^3}\left[25{,}000\pm\sqrt{25{,}000^2+42{,}000^2}\right]$$

$$=\frac{16}{\pi\times7\cdot5^3}\left[25{,}000\pm48{,}900\right]$$

i.e., $$f_1=\frac{16}{\pi\times7\cdot5^3}\times73{,}900=892\text{ kg/cm}^2$$

and $$f_2=-\frac{16}{\pi\times7\cdot5^3}\times23{,}900=-288\text{ kg/cm}^2$$

Maximum shearing stress $=\frac{1}{2}(f_1-f_2)=590$ kg/cm²

Simple tensile stress which will produce the same maximum shearing stress $=2\times590=1{,}180$ kg/cm²,

$$\text{Factor of safety}=\frac{3{,}800}{1{,}180}=3\cdot22.$$

7. A mild-steel hollow shaft of 10 cm external diameter and 5 cm internal diameter is subjected to a twisting moment of 80,000 kg cm and a bending moment of 25,000 kg cm. Poisson's ratio is $\frac{1}{4}$. Calculate the principal stresses and find the direct stress which, acting alone, would produce the same (a) maximum elastic strain energy, (b) maximum elastic shear strain energy, as that produced by the principal stresses acting together.

(Engineering Services, 1958)

$$\text{Principal stresses}=\frac{\frac{1}{2}[M\pm\sqrt{M^2+T^2}]}{I}\times\frac{D}{2}$$

$$= \frac{16D}{\pi(D^4 - d^4)}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$= \frac{16 \times 10}{\pi(10^4 - 5^4)}\left[25{,}000 \pm \sqrt{25{,}000^2 + 80{,}000^2}\right]$$

$$= \frac{16 \times 10}{\pi(10^4 - 5^4)}\left[25{,}000 \pm 83{,}800\right]$$

i.e., $$f_1 = \frac{16 \times 10}{\pi(10^4 - 5^4)} \times 108{,}800 = 591 \text{ kg/cm}^2$$

and $$f_2 = \frac{-16 \times 10}{\pi(10^4 - 5^4)} \times 58{,}800 = -319 \text{ kg/cm}^2$$

(a) Maximum strain energy :

$$f_1^2 + f_2^2 - \frac{2}{m} f_1 f_2 = f^2$$

or $$591^2 + 319^2 + \tfrac{2}{4} \times 591 \times 319 = f^2$$

$\therefore$ $$f = 738 \text{ kg/cm}^2.$$

(b) Maximum shear strain energy :

$$f_1^2 + f_2^2 - f_1 f_2 = f^2$$

or $$591^2 + 319^2 + 591 \times 319 = f^2$$

$\therefore$ $$f = 800 \text{ kg/cm}^2.$$

8. A circular shaft 10 cm diameter is subjected to combined bending and twisting moments, the bending moment being three times the twisting moment. If the direct tension yield point of the material is 3,600 kg/cm² and the factor of safety on yield is to be 4, calculate the allowable twisting moment by the following theories of elastic failure : (a) maximum principal stress, (b) maximum shearing stress, (c) maximum shear strain energy.

(Lond. Univ.)

$$\text{Principal stresses} = \frac{32}{\pi d^3} \times {}_2\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$= \frac{16}{\pi \times 10^3}\left[3T \pm \sqrt{9T^2 + T^2}\right]$$

$$= \frac{16T}{1{,}000\pi}\left[3 \pm \sqrt{10}\right]$$

i.e., $$f_1 = \frac{16T}{1{,}000\pi}\left[3 + \sqrt{10}\right]$$

and $$f_2=\frac{16T}{1{,}000\pi}\left[3-\sqrt{10}\right]$$

Allowable stress in simple tension

$$f=\frac{3{,}600}{4}=900 \text{ kg/cm}^2.$$

(a) Maximum principal stress theory :

$$\frac{16T}{1{,}000\pi}\left[3+\sqrt{10}\right]=900$$

$$\therefore \quad T=\mathbf{28{,}700 \text{ kg cm}}$$

(b) Maximum shearing stress theory :

f_2 is negative.

$$\therefore \quad \tfrac{1}{2}(f_1-f_2)=\frac{f}{2}$$

$$\text{or} \quad f_1-f_2=f$$

$$\text{or} \quad \frac{16T}{1{,}000\pi}\times 2\sqrt{10}=900$$

$$\therefore \quad T=\mathbf{27{,}900 \text{ kg cm}}$$

(c) Maximum shear strain energy theory :

$$f_1^2+f_2^2-f_1 f_2=f^2$$

$$\text{or} \quad \left(\frac{16T}{1{,}000\pi}\right)^2\times\left(9+10+6\sqrt{10}+9+10-6\sqrt{10}-9+10\right)=900^2$$

$$\text{or} \quad \left(\frac{16T}{1{,}000\pi}\right)^2\times 39=900^2$$

$$\therefore \quad T=\frac{900\times 1{,}000\pi}{16\times\sqrt{39}}=\mathbf{28{,}300 \text{ kg cm.}}$$

9. A direct-tension test on a specimen of steel gave elastic breakdown at 3,000 kg/cm². A shaft made of this material is 5 cm diameter. Determine according to the following theories the torque required to produce elastic breakdown when the shaft also carries a bending moment of 25,000 kg cm: (a) maximum principal stress theory, (b) maximum total strain energy theory. Poisson's ratio=0·3.

(Lond. Univ.)

$$\text{Principal stresses}=\frac{32}{\pi d^3}\times\frac{1}{2}\left[M\pm\sqrt{M^2+T^2}\right]$$

$$=\frac{16}{\pi\times 125}\left[25{,}000\pm\sqrt{25{,}000^2+T^2}\right]$$

i.e., $$f_1=\frac{16}{125\pi}\left[25{,}000+\sqrt{25{,}000^2+T^2}\right]$$

and $$f_2=\frac{16}{125\pi}\left[25{,}000-\sqrt{25{,}000^2+T^2}\right]$$

(a) Maximum principal stress theory :

$$\frac{16}{125\pi}\left[25{,}000+\sqrt{25{,}000^2+T^2}\right]=3{,}000$$

or $$25{,}000+\sqrt{25{,}000^2+T^2}=73{,}600$$

or $$25{,}000^2+T^2=48{,}600^2$$

$\therefore$ $$T=\mathbf{41{,}700\ kg\ cm.}$$

(b) Maximum total strain energy theory :

$$f_1^2+f_2^2-\frac{2}{m}f_1f_2=f^2$$

or $$\left(\frac{16}{125\pi}\right)^2\times\left(4\times 25{,}000^2+2{\cdot}6T^2\right)=3{,}000^2$$

or $$4\times 25{,}000^2+2{\cdot}6T^2=54{\cdot}2\times 10^8$$

or $$2{\cdot}6T^2=29{\cdot}2\times 10^8$$

$\therefore$ $$T=\mathbf{33{,}500\ kg\ cm.}$$

10. A specimen of steel has a yield-point stress in simple tension of 3,200 kg/cm². A shaft made of this material is 5 cm diameter and is subjected to a twisting moment of 25,000 kg cm. Assuming the criterion of elastic failure is the reaching of a definite value for the maximum shear strain energy per unit volume, calculate what additional bending moment will cause the material to pass the yield point.

$$\text{Principal stresses}=\frac{32}{\pi d^3}\times\tfrac{1}{2}[M\pm\sqrt{M^2+T^2}]$$

$$=\frac{16}{125\pi}[M\pm\sqrt{M^2+25{,}000^2}]$$

i.e. $$f_1=\frac{16}{125\pi}[M+\sqrt{M^2+25{,}000^2}]$$

and $$f_2=\frac{16}{125\pi}[M-\sqrt{M^2+25{,}000^2}]$$

$$f_1^2+f_2^2-f_1f_2=f^2$$

or $$\left(\frac{16}{125\pi}\right)^2\times[4M^2+3\times 25{,}000^2]=3{,}200^2$$

or $$4M^2+3\times 25{,}000^2=25{,}000^2\times\pi^2$$

or $$4M^2=6{\cdot}87\times 25{,}000^2$$

$$\therefore\quad M=32{,}800 \text{ kg cm.}$$

11. A hollow cylindrical brass beam, of inner and outer diameters 5 and 7·5 cm respectively, sustains on a certain cross-section a pure bending moment of 20,000 kg cm and an axial torque. If a factor of safety of 3 is required, what is the maximum torque that may be transmitted along the shaft if failure is reckoned to have occurred when the maximum shear strain energy per unit volume has reached a value corresponding to a simple tensile stress of 2,000 kg/cm² ?

(Lond. Univ.)

$$\text{Principal stresses}=\frac{D}{2I}\times\tfrac{1}{2}[M\pm\sqrt{M^2+T^2}\,]$$

$$=\frac{16D}{\pi(D^4-d^4)}\times[M\pm\sqrt{M^2+T^2}\,]$$

$$=\frac{16\times 7{\cdot}5}{\pi(7{\cdot}5^4-5^4)}\times[20{,}000\pm\sqrt{20{,}000^2+T^2}]$$

$$=\frac{120}{\pi(7{\cdot}5^4-5^4)}\times[20{,}000\pm\sqrt{20{,}000^2+T^2}]$$

i.e. $$f_1=\frac{120}{\pi(7{\cdot}5^4-5^4)}\times[20{,}000+\sqrt{20{,}000^2+T^2}]$$

and $$f_2=\frac{120}{\pi(7{\cdot}5^4-5^4)}\times[20{,}000-\sqrt{20{,}000^2+T^2}]$$

$$f_1^2+f_2^2-f_1f_2=f^2$$

or $$\left\{\frac{120}{\pi(7{\cdot}5^4-5^4)}\right\}^2\times(4\times 20{,}000^2+3T^2)=\left(\frac{2{,}000}{3}\right)^2$$

or $$4\times 20{,}000^2+3T^2=19{\cdot}65\times 10^8$$

or $$3T^2=3{\cdot}65\times 10^8$$

$$\therefore\quad T=11{,}030 \text{ kg cm.}$$

12. A torque of 15,000 kg cm is transmitted by a cylindrical tube 10 cm external diameter and of uniform thickness 0·25 cm. If the elastic limit of the material under simple tension is 2,500 kg/cm², calculate the factor of safety when the criterion of failure is (a) maximum shearing stress, (b) maximum shear strain energy.

(Lond. Univ.)

Maximum shear stress

$$q=\frac{T}{J}\cdot\frac{D}{2}=\frac{16TD}{\pi(D^4-d^4)}=\frac{16\times 15{,}000\times 10}{\pi(10^4-9\cdot5^4)}$$

$$=412 \text{ kg/cm}^2.$$

(a) Maximum shearing stress :

Simple tensile stress which will produce the same shearing stress

$$=2\times 412=824 \text{ kg/cm}^2$$

$\therefore$ Factor of safety $=\dfrac{2{,}500}{824}=3{\cdot}03.$

(b) Maximum shear strain energy :

Let f be the simple tensile stress which will produce the same shear strain energy.

Then $$\frac{q^2}{2C}=\frac{f^2}{6C}$$

or $$f=\sqrt{3}\times q=\sqrt{3}\times 412=714 \text{ kg/cm}^2$$

$\therefore$ Factor of safety $=\dfrac{2{,}500}{714}=3{\cdot}50$

13. A bending moment M applied to a solid round shaft causes a maximum direct stress f at elastic failure. Determine the numerical relationships between M and a twisting moment T which, acting alone on the shaft, will produce elastic failure, according to each of the following theories of failure: (a) maximum principal stress; (b) maximum principal strain; (c) maximum strain energy; (d) maximum shear stress; (e) maximum shear strain energy. Poisson's ratio $=0{\cdot}3$. (Lond. Univ.)

$$f=\frac{32M}{\pi d^3}$$

Due to torque T acting alone shearing stress

$$q=\frac{16T}{\pi d^3}$$

Principal stresses f_1 and $f_2=\pm q=\pm\dfrac{16T}{\pi d^3}$

(a) Maximum principal stress theory :

$$f_1=f$$

or $$\frac{16T}{\pi d^3}=\frac{32M}{\pi d^3}$$

$$\therefore\quad \mathrm{T}=2\,\mathrm{M}.$$

(b) Maximum principal strain theory :

$$\frac{1}{E}\left(f_1-\frac{f_2}{m}\right)=\frac{f}{E}$$

or $$\frac{16T}{\pi d^3}\left(1+\frac{1}{m}\right)=\frac{32M}{\pi d^3}$$

or $$T(1+0{\cdot}3)=2M$$

$$\therefore\quad \mathbf{T=1{\cdot}54\ M.}$$

(c) Maximum strain energy theory :

$$f_1^2+f_2^2-\frac{2}{m}f_1f_2=f^2$$

or $$\left(\frac{16T}{\pi d^3}\right)^2\left[1+1+\frac{2}{m}\right]=\left(\frac{32M}{\pi d^3}\right)^2$$

or $$T^2(2+0{\cdot}6)=4M^2$$

$$\therefore\quad \mathbf{T=1{\cdot}24\ M}$$

or

$$\frac{q^2}{2C}=\frac{f^2}{2E}$$

or $$\left(\frac{16T}{\pi d^3}\right)^2=\left(\frac{32M}{\pi d^3}\right)^2\times\frac{1}{2\left(1+\frac{1}{m}\right)}$$

or $$T^2=\frac{4M^2}{2\times 1{\cdot}3}$$

$$\therefore\quad T=1{\cdot}24M.$$

(d) Maximum shear stress theory :

$$q=\frac{f}{2}$$

or $$\frac{16T}{\pi d^3}=\frac{16M}{\pi d^3}$$

$$\therefore\quad \mathbf{T=M.}$$

(e) Maximum shear strain energy theory :

$$f_1^2+f_2^2-f_1f_2=f^2$$

or $$\left(\frac{16T}{\pi d^3}\right)^2[1+1+1]=\left(\frac{32M}{\pi d^3}\right)^2$$

or $$3T^2=4M^2$$

$$\therefore\quad \mathbf{T=1{\cdot}155\ M}$$

or

$$\frac{q^2}{2C} = \frac{f^2}{6C}$$

or $$\sqrt{3} \times q = f$$

or $$\sqrt{3} \times \frac{16T}{\pi d^3} = \frac{32M}{\pi d^3}$$

or $$\sqrt{3}T = 2M$$

$$\therefore \quad T = 1{\cdot}155 \text{ M.}$$

14. Three exactly similar specimens of mild-steel tube are 4 cm external diameter and 3·2 cm internal diameter. One of these is tested in tension and reaches the limit of proportionality at an axial tensile load of 9,000 kg. The second is tested in simple torsion. The third is also tested in torsion, but with a uniform bending moment of 3,500 kg cm applied throughout the test. Assuming maximum shear stress to be the criterion of elastic failure, estimate the torque at which the two torsion specimens should fail. (Lond. Univ.)

Under axial load :

$$f = \frac{9{,}000 \times 4}{\pi(4^2 - 3{\cdot}2^2)} = \frac{36{,}000}{5{\cdot}76\pi} \text{ kg/cm}^2$$

Under simple torsion :

$$q = \frac{T}{J} \cdot \frac{D}{2} = \frac{16TD}{\pi(D^4 - d^4)} = \frac{16T \times 4}{\pi(4^4 - 3{\cdot}2^4)}$$

$$= \frac{64T}{26{\cdot}24 \times 5{\cdot}76\pi}$$

$$q = \frac{f}{2}$$

or $$\frac{64T}{26{\cdot}24 \times 5{\cdot}76\pi} = \frac{18{,}000}{5{\cdot}76\pi}$$

$$\therefore \quad T = 7{,}380 \text{ kg cm}$$

Under combined bending and torsion :

Equivalent torque, $T_e = \sqrt{M^2 + T^2} = \sqrt{3{,}500^2 + T^2}$

Maximum shear stress

$$q = \frac{T_e}{J} \cdot \frac{D}{2} = \frac{64\sqrt{3{,}500^2 + T^2}}{26{\cdot}24 \times 5{\cdot}76\pi}$$

$$q = \frac{f}{2}$$

or $$\frac{64\times\sqrt{3500^2+T^2}}{26{\cdot}24\times5{\cdot}76\pi}=\frac{18{,}000}{5{\cdot}76\pi}$$

or $$3{,}500^2+T^2=7{,}380^2$$

$$\therefore\quad T=6{,}500 \text{ kg cm.}$$

15. A cylindrical shell made of mild-steel plate and 120 cm in diameter is to be subjected to an internal pressure of 15 kg/cm². If the material yields at 2,000 kg/cm², calculate the thickness of the plate on the basis of the following three theories, assuming a factor of safety 3 in each case. (a) Maximum principal stress theory, (b) Maximum shear stress theory, (c) Maximum shear strain energy theory.

(Treat it as a two-dimensional case neglecting 15 kg/cm² as a principal stress.) (Engineering Services, 1963)

Hoop stress $$f_1=\frac{pD}{2t}=\frac{15\times120}{2t}=\frac{900}{t}$$

Longitudinal stress $$f_2=\frac{pD}{4t}=\frac{450}{t}$$

Premissible stress in simple tension

$$f=\frac{2{,}000}{3}\text{ kg/cm}^2$$

(a) Maximum principal stress theory :

$$\frac{900}{t}=\frac{2{,}000}{3}$$

$$\therefore\quad t=1{\cdot}35\text{ cm.}$$

(b) Maximum shear stress theory :

$$\frac{f_1}{2}=\frac{f}{2}$$

or $$f_1=f$$

or $$\frac{900}{t}=\frac{2{,}000}{3}$$

$$\therefore\quad t=1{\cdot}35\text{ cm.}$$

(c) Maximum shear strain energy theory :

$$f_1^2+f_2^2-f_1f_2=f^2$$

or $$\left(\frac{900}{t}\right)^2+\left(\frac{450}{t}\right)^2-\left(\frac{900}{t}\right)\times\left(\frac{450}{t}\right)=\left(\frac{2{,}000}{3}\right)^2$$

or $$3\times\left(\frac{450}{t}\right)^2=\left(\frac{2{,}000}{3}\right)^2$$

$$\therefore \quad t = \frac{\sqrt{3} \times 450 \times 3}{2{,}000} = 1{\cdot}169 \text{ cm.}$$

16. A cylindrical drum of 60 cm internal diameter is to withstand an internal pressure of 20 kg/cm². Calculate the necessary wall thickness for a factor of safety of 3 if the criterion of failure is the maximum strain energy, and the elastic limit in pure tension is 2,400 kg/cm². Poisson's ratio = 0·3.

(Lond. Univ.)

(Ans. 0·731 cm)

17. A long straight tube is subjected to an internal pressure of 70 kg/cm². The internal diameter of the tube is 9 cm and the walls of the tube are 3 mm thick. Considering the tube as a thin cylinder, determine the longitudinal and circumferential stresses.

This tube is then subjected to a twisting moment of 3,000 kg cm. Find the factor of safety according to (a) the maximum principal stress theory; (b) the maximum strain theory; and (c) the maximum shear stress theory. Assume that the Poisson's ratio = 0·3; elastic limit stress = 2,800 kg/cm²; $E = 2 \times 10^6$ kg/cm². (Engineering Services, 1960)

Hoop stress, $$f_x = \frac{pD}{2t} = \frac{70 \times 9}{2 \times 0{\cdot}3} = 1{,}050 \text{ kg/cm}^2$$

Longitudinal stress, $$f_y = \frac{pD}{4t} = 525 \text{ kg/cm}^2$$

Shear stress due to torque,

$$q = \frac{\text{torque}}{\text{cross-sectional area} \times \text{radius}}$$

$$= \frac{3{,}000}{(\pi \times 9 \times 0{\cdot}3) \times 4{\cdot}5}$$

$$= 78{\cdot}6 \text{ kg/cm}^2$$

$$\text{Principal stresses} = \tfrac{1}{2}(f_x + f_y) \pm \tfrac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

$$= \tfrac{1}{2} \times 1{,}575 \pm \tfrac{1}{2}\sqrt{525^2 + 4 \times 78{\cdot}6^2}$$

$$= 787{\cdot}5 \pm 274$$

i.e., $f_1 = 1061{\cdot}5$ kg/cm² and $f_2 = 513{\cdot}5$ kg/cm²

(a) Maximum principal stress theory:

Factor of safety $$= \frac{2{,}800}{1061{\cdot}5} = 2{\cdot}64.$$

(b) Maximnm strain theory:

Maximum strain $$= \frac{1}{E}\left(f_1 - \frac{f_2}{m}\right)$$

$$=\frac{1}{E}(1061{\cdot}5-0{\cdot}3\times 513{\cdot}5)=\frac{907{\cdot}5}{E}$$

Simple stress to produce this strain $=907{\cdot}5$ kg/cm²

$\therefore$ Factor of safety $=\dfrac{2{,}800}{907{\cdot}5}=3{\cdot}09.$

(c) Maximum shear stress theory :

Maximum shear stress $=\dfrac{f_1}{2}=530{\cdot}75$ kg/cm²

Simple tensile stress to produce the same maximum shear stress

$$=530{\cdot}75\times 2=1061{\cdot}5 \text{ kg/cm}^2$$

$\therefore$ Factor of safety $=\dfrac{2{,}800}{1061{\cdot}5}=2{\cdot}64.$

18. A thin cylinder of internal diameter 30 cm, with closed ends, is subjected to an internal pressure of 40 kg/cm² and to an axial torque of 7,000 kg m. The maximum safe stress for the material in simple tension is 1,600 kg/cm². Find the thickness for the cylinder wall if the maximum shear strain energy is the criterion of failure.

Hoop stress, $f_x=\dfrac{pD}{2t}=\dfrac{40\times 30}{2t}=\dfrac{600}{t}$

Longitudinal stress, $f_y=\dfrac{pD}{4t}=\dfrac{300}{t}$

Shear stress $q=\dfrac{7{,}000\times 100}{(\pi\times 30\times t)\times 15}=\dfrac{495}{t}$

Principal stresses

$$=\tfrac{1}{2}\left(\frac{600}{t}+\frac{300}{t}\right)\pm\tfrac{1}{2}\sqrt{\left(\frac{600}{t}-\frac{300}{t}\right)^2+4\times\left(\frac{495}{t}\right)^2}$$

$$=\frac{450}{t}\pm\frac{517}{t}$$

i.e. $f_1=\dfrac{967}{t}$ and $f_2=-\dfrac{67}{t}$

$$f_1^2+f_2^2-f_1f_2=f^2$$

or $$\left(\frac{967}{t}\right)^2+\left(\frac{67}{t}\right)^2+\left(\frac{967}{t}\right)\times\left(\frac{67}{t}\right)=1,600^2$$

or $$\frac{1,004,000}{t^2}=1,600^2$$

$$\therefore \quad t=\frac{1,002}{1,600}=0\cdot626 \text{ cm.}$$

19. A cast-iron cylinder has outside and inside diameters of 20 cm and 12 cm. If the ultimate tensile strength of the cast-iron is 1,500 kg/cm² find, according to each of the following theories of failure, the internal pressure which would cause rupture: (a) Maximum principal stress theory, (b) Maximum strain theory, (c) Maximum strain energy theory. Assume no longitudinal stress in the cylinder. Poisson's ratio=0·25.

(Lond. Univ.)

Greatest circumferential stress at the inside surface

$$f_1=\frac{R_1^2+R_2^2}{R_1^2-R_2^2}\cdot p=\frac{100+36}{100-36}\cdot p=\frac{17}{8}p$$

The other principal stress at the inside, $f_2=-p$

(a) Maximum principal stress theory :

$$\frac{17}{8}p=1,500$$

$$\therefore \quad p=706 \text{ kg/cm}^2.$$

(b) Maximum strain theory :

$$\frac{1}{E}\left(f_1-\frac{f_2}{m}\right)=\frac{1,500}{E}$$

or $$\frac{17}{8}p+0\cdot25p=1,500$$

$$\therefore \quad p=632 \text{ kg/cm}^2.$$

(c) Maximum strain energy theory :

$$f_1^2+f_2^2-\frac{2}{m}f_1f_2=f^2$$

or $$(\tfrac{17}{8}p)^2+p^2+2\times0\cdot25\times\tfrac{17}{8}p\times p=1,500^2$$

or $$\frac{421}{64}p^2=1,500^2$$

$$\therefore \quad p=\frac{1,500\times8}{\sqrt{421}}=585 \text{ kg/cm}^2.$$

20. A shaft 10 cm in diameter transmits a torque of 130,000 kg cm together with an axial thrust. Find the maximum value of this thrust which will give a factor of safety of 2 against elastic failure by (a) maximum shearing stress; (b) maximum shear strain energy. The elastic limit of the material is 3,000 kg/cm². (Lond. Univ.)

Permissible direct stress, $f = \dfrac{3{,}000}{2} = 1{,}500$ kg/cm²

Shear stress due to torque,

$$q = \frac{16T}{\pi d^3} = \frac{16 \times 130{,}000}{\pi \times 10^3} = 662 \text{ kg/cm}^2$$

Let f_x be the compressive stress due to axial thrust.

$$\text{Principal stresses} = \tfrac{1}{2}f_x \pm \tfrac{1}{2}\sqrt{f_x^2 + 4q^2}$$

$$= \tfrac{1}{2}f_x \pm \tfrac{1}{2}\sqrt{f_x^2 + 4 \times 662^2}$$

i.e., $$f_1 = \tfrac{1}{2}f_x + \tfrac{1}{2}\sqrt{f_x^2 + 4 \times 662^2}$$

and $$f_2 = \tfrac{1}{2}f_x - \tfrac{1}{2}\sqrt{f_x^2 + 4 \times 662^2}$$

(a) Maximum shearing stress theory :

f_2 is negative.

$\therefore$ $$\tfrac{1}{2}(f_1 - f_2) = \tfrac{1}{2}f$$

or $$f_1 - f_2 = f$$

or $$\sqrt{f_x^2 + 4 \times 662^2} = 1{,}500$$

or $$f_x^2 + 4 \times 662^2 = 1{,}500^2$$

$\therefore$ $$f_x = 705 \text{ kg/cm}^2$$

$\therefore$ $$P = \frac{\pi}{4} \times 10^2 \times 705 = \mathbf{55{,}400 \text{ kg.}}$$

(b) Maximum shear strain energy

$$f_1^2 + f_2^2 - f_1 f_2 = f^2$$

or $$\frac{f_x^2}{4} + \frac{1}{4}\left(f_x^2 + 4 \times 662^2\right) + \frac{f_x^2}{4} + \frac{1}{4}\left(f_x^2 + 4 \times 662^2\right)$$
$$- \frac{f_x^2}{4} + \frac{1}{4}\left(f_x^2 + 4 \times 662^2\right) = 1{,}500^2$$

or $$f_x^2 + 3 \times 662^2 = 1{,}500^2$$

or $$f_x = 967 \text{ kg/cm}^2$$

$$P = \frac{\pi}{4} \times 10^2 \times 967 = \mathbf{75{,}900 \text{ kg.}}$$

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